Can points, lines, planes and so forth be subjected directly to calculation? That has seldom been done so far even tho Hermann Grassmann showed in 1844 how "*inexhaustibly fruitful*" it actually is. Its key is that *points are numbers*—***geometric numbers***—and their extension generates lines, planes and on up, *which are also numbers*. Grassmann's path to that idea was so esoteric and meandering that few have untangled it over the last two centuries. Surprisingly, it can easily be untangled by anyone willing to replace the *often-discussed* dimension of a point with the *seldom-articulated* one. There's the rub: our own path toward that dimension has been as meandering and baffling as Grassmann's path was. To untangle both paths we shall have to go back several millennia and proceed straight and narrow. This will be quick: we will be flying low over the winding river of ideas leading toward *formal geometric dimension*.

Meandering toward Geometric Algebra

Gary Harper

Formal geometric dimension is the starring enabler in this narrowly focused journey. One might suppose that we humans, after several thousand years attempting to formalize dimension, would now understand it well. In fact we moderns, with some recent encouraging grade-articulating exceptions, have actually wandered farther from *geometric* dimension than our forebears. We have instead embraced *syntactic* dimension: the *n*-size of a carton of scalars, denoted as $\boldsymbol{R}^n$.

That abstract notion would have bemused the ancients, who carefully partitioned scalars into *Numbers* (*natural* ones, meaning positive integers) and *Magnitudes*. Magnitudes themselves they carefully partitioned into distinct kinds: *length, area, volume*, subsequently assigned numeric dimensions 1, 2, 3.

Under that assignment, an ancient might have wondered: How does amalgamating the integers with dimension-ignoring magnitudes, and then *n*-boxing them in, suddenly start paying attention to *n*-dimension? Wouldn't *n* greater than three be impossible? And what about those boxes of $\boldsymbol{C}^n$ so-called "scalars" composed of not just integers and inept magnitudes, but also "imaginaries"? Does that even-more-mixed concoction pay attention to *n*-dimension too? … and so on. Their imagined bemusements about our abstractions would certainly be more severe than our real bemusements about their particularities.

Their bemusements might well have been forestalled when particular ***numbers*** *with dimensioned locus* appeared a century prior to our recent *n*-box scalar abstractions. The originator of those numbers, Hermann Grassmann, promptly recognized in the foreword to his first book that *endlessly many* of them—variously oriented patches of plane—express imaginary i. So, "*all imaginary expressions now acquire a* ***purely geometric*** *meaning*".[1] Unfortunately, he burdened his new numbers with timeworn dimensional inconsistencies that he—and we—inherited from Euclid; so we have not grasped them firmly yet.

To do so at last, we must at first grasp Euclid's dimensional ideas, *firmly*. Many of them had been haphazardly generated by the mystic Pythagoreans. They had come to believe, oft told, that *All is Number* after discovering that musical harmonies correspond to ratios of integers. When they tried to apply that simplifying idea to their fabled Pythagorean Theorem they discovered, contrariwise, that some triangle sides cannot be expressed by such ratios. That unexpected complication led to a drowning, so the various stories go, either distraught or malicious. Subsequent thinkers came to believe that *Most is geometric Locus and Magnitude, All the rest is Number*.

Euclid inherited those ideas, so the ancient *Locus-Magnitude-Number* trifurcation became engraved into his thirteen "books"; short chapters really, each of whose understanding does require more effort than any ordinary book would. Investing that effort is an excellent way to acquire the habit of precise thought; and self-schooling Abe Lincoln began to do just that. As he set out he encountered one intricate figure after another, each one building on the preceding figures. But then suddenly, at book V, the intricate figures gave way to page after page of segmented parallel lines, a vista that must have appeared refreshingly simple to young Abe. He had arrived, unsuspecting, at Euclid's disjunction between *Locus* and *Magnitude*. That is where he stopped, never penetrating beyond to *Number* in Books VII, VIII, IX.

You might have stopped at *Magnitude* too unless someone had advised you about the peculiar encumbrances the ancients imposed on it. Euclid isn't going to: he declined to deal with *Encumbrances*, and instead confined himself exclusively to *Propositions* derived from *Postulates* and *Common Notions*, augmented by *Definitions*. To discover *Magnitude Encumbrances* you have to read between Euclid's lines, for which Definition 5 in Book V is a perfect starting point:

> Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples taken in corresponding order.[2]

Got that? I am betting that this not-so-refreshingly-simple definition is what stopped Abe. Definitions 1 thru 4—*part*, *multiple*, *ratio* and *have a ratio*—were simple but nuanced. The nuance is that they all dealt with geometric magnitudes: *length, area, volume*. Euclid illustrated such magnitudes in Book V with what he believed were the simplest kind of them: lengths of line segments, which he knew to be either *commensurate* or *incommensurate*—they may be measurable by the same unit or not. This was his Pythagorean heritage, a *Fundamental Astonishment* in his era like Relativity or Quantum Theory are in our own.

Lengths propagate that complication up to areas and volumes, ideas that Euclid extensively elaborates in Book X, which consumes nearly a fifth of his work. His goal here in Book V, unannounced but *very* prominent in his mind, is to start confronting the astonishment at what he (mistakenly) considered its lowest dimension.

Definition 5 tacitly confronts the *Magnitude* astonishment in a cunning way: In modern terms it effectively traps a real number between rational ones, or on one of them. In ancient terms there are dimensional complications: it implicitly defines a *Magnitude* ratio *of the same kind*—*length/length*, *area/area*, *volume/volume*—via ratios of natural *Number*s, called *equimultiples* here and elsewhere. That ploy may seem to recover the *All-is-Number* idea, but it would not likely have been acceptable to any Pythagorean: Euclid's two *whatever*'s make the process infinite, typically. Only when the same-dimensioned magnitudes are commensurate would their ratio get trapped finitely on a rational number. This understated definition I find astonishingly clever—*Dedekind* clever.

The astonishment for modern readers of Book V, however, is more likely to be how transparent and straightforward its opaque Propositions become under a simple switch from *discussing them logically via postulates* to actually *articulating them algebraically via rules*: translate each *Magnitude* proposition into an equation, or several usually, and then manipulate them with elementary algebra, being careful not to mix dimensions.

It is all too easy for us to criticize Euclid's *Magnitude* discussion now that we have finally acquired *simple algebraic tools* that articulate them crisply. The simplicity of our tools belies the prolonged fortuity needed to come up with them: not until the late 1500s did symbolic equations became available to manipulate real numbers. Even then the equals sign itself was still not available. It arose only after "*equals*" had become so pervasive that it was profitably replaced by "=", quickly adopted.

So why did it take *nearly two millennia* to achieve symbolism that in retrospect seems so obvious? This is the conundrum with mathematical evolution: whereas human cleverness with intellectual tools within reach is common—and often astonishing indeed—forging better tools is not common. Grassmann was uncommon in both ways: he was not notably clever with extant algebraic tools;[3] he instead replaced them with *simple geometric tools*.

His new tools shall induce further astonishment about Euclid's work when we finally grasp them: Not only do Euclid's *Magnitude* propositions become transparent under a switch from discussion to articulation, so too do his *Locus* propositions. The Pythagorean Theorem for example, Proposition 47 Book I, becomes a straightforward calculation.

Unfortunately, it is one of the few *Locus* propositions we can articulate now: it does not depend on any *particular* locus, and we have recently acquired the *free* sublanguage of Grassmann's algebra, at least many of us have. What we can't articulate yet are Euclid's propositions that do depend on particular locus—we have not yet acquired Grassmann's ***bound*** foundation for his algebra, at least most of us haven't.

Our blinders began to develop when the ancients burdened *Magnitudes* with dimensional complications. They had the right idea: *something* must make dimensional distinctions, but they chose the wrong thing to do it—*something with no locus*. What they needed was ***something with locus***, *primitive* locus. They did get an incidental grasp on it; but by discussing it rather than articulating it, they accidentally removed its ability to interact with its geometric kin. This rendered it incapable of generating dimensional distinctions, a problem that persists even now, here at the launch of the Third Millennium, Common Era.

The problem crystallized some twenty-three hundred years ago with the very first sentence in Euclid's work, translated literally by Thomas Heath like this:

A *point* is that which has no part.[p1]

This was translated meaningfully by John Playfair like so:

A *point* is that which has position, but not magnitude.[4]

Playfair's translation parallels his next one, at least conceptually: "A *line* is length without breadth", more literally translated by Heath as "A *line* is breadthless length." Heath's translations may seem vague; Playfair's may seem cavalier; but neither is true.

That *a point has no part* seems to be the vague truism that a point has no spatial expanse like other geometric elements have. Euclid knew that of course, but that was not what he was saying (tho he likely intended it also): *part*, like most of his vocabulary, is a technical term, whose definition he deferred to Book V:

1. A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

To understand that you have to understand what Euclid means by *measures*, an idea clarified by his next definition:

2. The greater is a *multiple* of the less when it is measured by the less.

Putting 1 and 2 together, *a part is an integral subdivision of magnitude*, and—going right back to the beginning—*points have none* according to Euclid. This effectively denies points magnitude and justifies Playfair's translation, which is as careful about meaning as Heath's is about literal vocabulary.

It is easy—and it has come to seem natural after several millennia—to deny points magnitude when you are discussing them logically. Logical discussion leaves you free to define things however you want, provided your definitions mesh with each other and do not introduce unwarranted assumptions. The danger is that you might do so and define away the ability to mesh at all, which is just what Euclid did to points: in denying them magnitude, he denied them ability to mesh with things he endowed with magnitude.

This only becomes apparent, however, when you try to actually *articulate* points, an endeavor few have tried so far: most notably Mobius, Grassmann, Clifford, Peano, Whitehead, spanning the 1800s. They were all forced to give points magnitude, which they called *weight*. So *weight* is actually the *simplest* kind—the *lowest-dimensioned* kind—of magnitude, not *length*; but such has been Euclid's tremendous intellectual inertia that this inherently contra-Euclidian idea remains largely unnoticed. Mobius's classic 1827 *Barycentric Calculus*, for example, has not yet been published in an English translation. Consequently, many mathematicians still assert that you can't even add points, saying silly things like you can't add London to Paris. Not only can you *add* them to produce a balance point, but you can also *multiply* them, as the just-mentioned luminaries showed (you'll see how, exactly). ***Points are numbers***, *geometric numbers*, and they need magnitude in order to obey the rules numbers obey.

This does not mean points have spatial expanse like their geometric kin; it merely means they have scaling relations with their kin. Scaling relations and spatial expanse are quite different. Scaling is a syntactic property disciplined by the rules; spatial expanse is a semantic property established by *formal geometric dimension*.

Points must have such dimension because they are geometrically *primitive*, meaning that they provide the foundational dimension for other geometric things. When Euclid's lengths, areas, volumes were assigned ad hoc numeric dimensions 1, 2, 3—apparently natural, obviously right—the only dimension left for points was 0, *meaning no dimension at all*. This works fine if points don't have magnitude, or need it, as most people still believe. It doesn't work at all when points do need magnitude in order to generate other geometric magnitudes. Points must then have primitive dimension **1**; otherwise they couldn't generate anything.

Does this imply that lines, areas, volumes have dimensions **2**, **3**, **4**? Yes it does: count the independent[5] ***points*** needed to *generate* those things. Such generation makes them as **bound** as points are (denoted in **bold**). In startling contrast, generating *free* "lines", "areas", "volumes" gives them *intrinsically composite* dimensions {**1**, **1**}, {**2**, **2**}, {**3**, **3**} (you'll also see this, exactly), usefully abbreviated to 1, 2, 3 (non-bold). Those abbreviations make Euclid's dimensions less than half right—more than half wrong. They leave dimension 0 for *scalars*, which works fine if you want them to become bona fide *geometric numbers* like points are. Such expressive inclusion makes scalars unique in *having no dimension or locus*, unlike ***points***, which ***do have dimension and locus***.

Attempting now to demystify the preceding *formal geometric dimensions* might make geometric numbers easier to grasp when you encounter them shortly. At the very least it should give some understanding of just how inhibiting Euclid's communal misconceptions have been over the past several thousand years—they have inhibited us from articulating points. Finally doing so generates utterly unexpected distinctions, empowering ones. To get to them we must trudge thru several millennia of expected distinctions. This will go fast if we focus resolutely on dimension, and on efforts to stop discussing and start articulating.

Diophantus started us on that path at the dawn of the Common Era by abbreviating his technical vocabulary, but he sidestepped dimensional complications by avoiding *Magnitude* and focusing on *Number*. Unfortunately, his syncopated algebra was not adopted by his early successors, whose dimensional efforts were typically manifested with explorations involving conic sections.

Conics, it became apparent, can be expressed by certain polynomials; so efforts to articulate *Magnitude* eventually focused on polynomials. Those efforts culminated in acrimonious public competitions staged in the early 1500s to solve cubic polynomials.

Solutions involved square roots and cube roots, which began to erode the ancient split between *Magnitude* and *Number*: roots of *Number*s are often irrational like *Magnitude*s often are; so the split between *Number* and *Magnitude* began to fuse into just plain number, what we now call *real number*. As that unification began to take hold, the ancient dimensional complications attached to *Magnitude* began to evaporate. The contestants, for example, were more concerned with the expression of their solutions than with the (volumetric) dimension of their problems, tho they did consider it clumsily.

In retrospect it would have been fortunate if mathematicians had considered dimension a little longer: root solutions induced new numbers with a *different dimension*, still little recognized. Here is the start of that story:

Solutions often generated square roots of negative numbers, reflexively rejected by the contestants, but not by their promoter and eventual explicator, Girolamo Cardano. He thought such roots might be used as validly as negative numbers.[6] Subsequent mathematicians concurred, showing that whereas rejection of those roots might be acceptable for quadratic solutions, it was not acceptable for cubic ones, which always have at least one real root. Solutions of cubics often generate square roots of negative numbers, which sometimes collaborate with each other to produce a real root.

Consequently, square roots of negatives *slowly* became legitimate. This launched the *pragmatic mysticism*[7] phase of mathematics: those roots work even tho they mystified their wielders. Four centuries later we realize that they work *very* well; so we have finally convinced ourselves that they don't mystify us anymore—we are now well into the fruitful *plastered-over mysticism* phase of mathematics: square roots of negatives have become our everyday "imaginary scalars", apparently having the same dimension as any other obeyers of the field axioms. They actually have different dimensions, as Grassmann began to understand long ago, as aficionados of Geometric Algebra understand now. Few others have even considered that idea because dimensioned scalars were discarded in the 1600s.

The last mathematician to take them seriously was Francois Viete in the late 1500s. He is the one who finally stopped discussing and started articulating; he is the one who made available symbolic equations to manipulate real numbers. He achieved that by studying the ancient Greeks with the express purpose of formalizing their ideas. His efforts culminated in an elegant advance and a clumsy regression, fortunately ignored.

His advance was his far-ranging enhancement of Diophantus, a primary Greek hero of his. Not only did Viete abbreviate his technical vocabulary like Diophantus had, but he went even further by allowing letters to stand in for *completely arbitrary* numbers. This very expressive ploy amounts to crisp articulation—for the first time in history—of Euclid's many verbose *whatever* discussions about *Magnitude*.

Then Viete went still further by designating his unknowns by vowels, his knowns by consonants. That apparently trivial innovation, in his hands, had dramatic consequences: it allowed him to convert an ancient analytic ploy into a systematic equation-solving strategy. The ancient ploy he described as "*the assumption of the thing sought as if admitted [and the arrival] by consequences at something truly admitted*."[8] By fixing your gaze on one of Viete's now-well-delineated unknowns, *as if admitted*, you can systematically isolate it on one side of an equation to arrive *at something truly admitted* by the knowns on the other side, as every good algebra student knows. This simple tactic is unexpectedly effective, and has become a vital part of our intellectual atmosphere. We should be honoring it as *Vietizing* a solution, or *Viete's Un-Known Stratagem*, or somesuch.

Even more wonderful, Viete's letters do not just represent arbitrary real numbers, they actually represent any "*species*" of things that obey his rules. In other words, *Viete's rules now define his objects*, rather than the other way around, as had been the case before he arrived on the scene. This means that his objects don't need to be real numbers (tho they usually were); *they can be any things that obey his rules*, ***like points for example***. Viete didn't get *that* wonderful; and, with a few curious exceptions, neither have we.

Any one of Viete's innovations would have been a tremendous advance during the several thousand years preceding him, but he compressed all of them into the last quarter of the 1500s. It is as if, when humans began to slowly disperse out of Africa, one of them suddenly went to the moon.

On the way, as you might expect, he stumbled a little: He enhanced the ancient Greek idea of *implicitly* burdening real numbers with dimensional distinctions by now doing it *explicitly*—he was merely continuing to formalize their informalities. Here, for example, is an anglicized version of his generic quadratic equation:

$$B * A\mathit{sq} + C\mathit{p} * A \text{ equals } D\mathit{s}$$

Vowel A is his unknown here; A*sq* is our A^2 (Descartes' innovation); but what are *p* and *s*? They are *dimensional appendages* to C and D, *plane* and *solid*, intended to enforce *dimensional homogeneity*: Viete, like his ancient Greek heroes, only allows *Number*s to be added to *Number*s, lengths to lengths, areas to areas; and of course volumes to volumes, as here. This ensures that his dimensioned sums are non-composite: they produce a singular result with a singular dimension, or none at all in the case of *Number*.

There are two problems with this: First, it hobbles his algebra: an articulate *geometric product* (coming up) generates *intrinsically composite sums*, from which you may extract pertinent dimensions. Of greater immediate relevance, however, *intrinsically composite sums are crucial for the free versus **bound** distinction*, as you'll see. The second problem is that an articulate algebra attaches dimension to those things that actually have it, rather than to scalars, which lack it. Fortunately, few paid any attention to Viete's dimensional appendages, not even the "second Viete", Pierre Fermat.[9]

Viete's well-honored successor, Rene Descartes, got rid of dimensional appendages—and indeed dimensional *anything*—in a much-celebrated strategy often described as "*the reduction of geometry to number*". He explains it in the very first sentence in his wildly successful book on geometry:

> Any problem in geometry can easily be reduced to such terms that a knowledge of lengths of certain straight lines is sufficient for its construction.[10]

That pedestrian statement does not do justice to the radical novelty he has just introduced. Your foreknowledge of *formal geometric dimension* can do it justice like so:

Dimension in geometry can be discarded, making things easy to calculate.

In other words, by uniformly reducing geometry to *lengths of lines*, Descartes has descended to scalars, *which have no dimension*, and are easy to calculate with. Here is his generative tactic:

> We first suppose the solution already effected, and give names for all lines that seem needful for its construction—to those that are unknown as well as those that are known.[p6]

Does that sound familiar? It is Descartes' geometricized version of Viete's algebraic *analysis*. Descartes' *synthesis*, by contrast, is radically different from any *anyone*'s previous synthesis, in which a person would proceed from analytic truths to their logical consequences. Viete, and indeed everyone prior to him, would have proceeded from the various real numbers uncovered by his analysis back up to lengths, areas, volumes.

Descartes, in stark contrast, *having annihilated dimension, never lets it reemerge*. He deplored the ancient dimensioned terms *square* and *cube:*[11] "*by a^2, b^3 and similar expressions I ordinarily mean only simple [**lengths of**] lines, which, however, I name squares, cubes, etc., so that I may make use of terms [**injudiciously**] employed in algebra.*"[p5] So when Descartes operates on scalars *he always gets other scalars*. He never gets areas or volumes like his intellectual forbears would have.

Here is how he achieves his dimensionlessness: Once he has given *geometric* names to the lines needful to him, he gives *algebraic* names to their ***scalar lengths***. *Then he remains within scalar algebra*, never ascending back to geometry. Let him explain: "*Thus to add lines BD and GH, I call one a, the other b, and write a + b*." Now he can compute $a + b$, which is of course another scalar. But so is ab, a/b, a^2, a^3, $\sqrt{(a^2 + b^2)}$, $\sqrt[3]{(a^3 - b^3 + ab^2)}$ "*and so on indefinitely*."[p5] His figures expose his computational strategy: he uses similar triangles and the Pythagorean Theorem to descend to scalars and—once down there—to also calculate with them. Such a tenacious real-number ploy would not startle most of us (which testifies to his wild success) but it certainly would have startled his predecessors.

There is something, however, that would startle most of us: Descartes almost never uses so-called "*Cartesian coordinates*"! The "*lines that seem needful*" for his constructions were *very* seldom normalized and mutually orthogonal ones. They were typically a jumbled haystack of lines augmented by the odd circle or two; and the various parts often moved in skewed directions at different rates. So orthonormal coordinates, what most people consider Descartes' revolutionary breakthru, were usually too trivial for his purposes. His actual breakthru—***dispensing with dimension*** *to make calculation easy*—is seldom stated so forthrightly because the *formal geometric dimension* of scalars seldom is.

Dispensing with dimension, altho it has proved computationally fecund, does have a problem: how do you get dimension back? The solution our mathematical community has devised is to box up dimensionless things. Stranger still, our community has even begun to box up *dimensioned things* that obey the same rules dimensionless things obey, like complex numbers for example. That is the ultimate outcome of overly celebrated ***reducing*** *geometry to number*. Happily, there is a *much* better way to get dimension back, still woefully uncelebrated: don't ever lose it—***enhance*** *geometry to number*:

> In extension theory there appears a characteristic method of calculation which, transposed into geometry, is inexhaustibly fruitful and consists in subjecting spatial structures (points, lines and so forth) directly to calculation.[12]

Grassmann wrote this in 1845 in response to an editorial request to clarify ideas in his 1844 book. That book initiated a *dimensioned* fusion of the ancient *Locus-Magnitude-Number* trifurcation into all-encompassing ***geometric number***; a fusion *bafflingly overdue*. When that idea does at last take hold it will supersede Descartes' *de-dimensioned* fusion: final clinching of the *Magnitude-Number* bifurcation into encompassing ***real number***, a fusion *promptly adopted*. Sadly, Grassmann's *enhancing* path—unlike Descartes' *reducing* path—was meandering, esoteric, and largely viewed as the work of an interloper.

One might need, as I needed, twenty years of effort to untangle it, with uncertain promise of success. Happily, his actual calculations—once he had finally happened onto the **1**-dimension of points—are so straightforward that you can easily re-create them for yourself, like so:

Start with a point, which you now realize has dimension **1**, an empowering discovery. Extend it to another point, producing a directed line segment, dimension **2**. The rules of your new algebra will allow this line segment to move anywhere within the line thru itself, *but only there*. So this is a ***bound*** vector, a ***great surprise*** to Grassmann, who had initially "*proved*" that it is free to move anywhere parallel to itself.[13] You might have thought the same thing yourself, initially, like I did.

Naturally you can keep going: extend your bound vector to yet another point, thereby generating a bound bivector, a directed parallelogram, dimension **3**, able to move anywhere within the plane thru itself, but only there. It has a fixed (area) magnitude but no particular shape owing to the constrained freedom of bondage. Extend that to another point, generating a bound trivector, a directed parallelepiped, whose properties you can surmise. And so on up beyond physical space, one extra dimension at a time.

And where does that extra dimension keep coming from? From the primitive dimension of a point of course, a successively *outer* one. That is why Euclid's traditional weightless and dimensionless points cannot be articulated.

Grassmann's extension generates a *multitude of points*; so it is a literal *multiplication* of them, which he called the *outer product*. It distributes over addition and scaling like ordinary multiplication, but it *neg*-commutes:[14] ***a* To *b*** = – (***b* To *a***), which is obvious geometrically. So points don't quite obey the field axioms that imaginary numbers do; and yet their neg-commuting ultimately generates imaginary numbers, an endless supply in fact, of various dimensions. Here is how: *spatial* neg-commuting generates *quarter-turners*: *right-facers* or *left-facers*—***ortho-turners***—***whose square is*** *a half turn*: an *about-face*, ***multiplication by minus one*** (how imaginary is that?).

That is just the start of spatial neg-commuting's remarkable fertility. To watch it begin spawning other things, shift your gaze to a different kind of neg-commuting, another *spatial* one, subtraction of two points:

$$\boldsymbol{a}-\boldsymbol{b} = -(\boldsymbol{b}-\boldsymbol{a}).$$

This is *free vector*, a parallel-roving bundle of two points, dimension {**1**, **1**}, usefully abbreviated to 1. It parallel-roves because the result of this addition diminishes to infinity in a certain direction, but never actually arrives there (nothing would be left). So this addition is *intrinsically composite*: its result is not well defined (it is a geometric kind of 0/0); only its summands are. There are endlessly many such point summands that diminish to infinity in exactly the same way, meaning that *they are all equal*. They all have the same direction and the same *separation*,[15] but different locations. Such *formal freedom* surprised Grassmann as much as his ***formal bondage*** had. (Both astonished me.)

Having generated composite free vectors, you can extend *them* like you did points: Extend free vector ***a–b*** to free-vector ***c–d***, say, making sure to obey the just-mentioned rules. You get a free bivector—give it a try.[16] You will discover that it is an intrinsically composite bundle of two separate but otherwise exactly opposite bound vectors, dimension {**2**, **2**}, usefully shortened to 2. It has a fixed (area) separation and fixed (planar) direction, but no particular shape or location—it is as *parallel-roving* as the free vectors that generated it. Extend it to another free vector. You get a free trivector, whose *extensive* properties (in both senses) you can surmise. And so on up, one extra dimension at a time.

If this story has seemed strange so far, it is now going to seem *really* strange. Grassmann eventually composed all of these distinctions; and his mature manipulation of them leaves little doubt that he understood them very well indeed. In particular, he understood that, whereas extension generates extra dimension, addition does not, at least *formally*. Nevertheless, under the influence of Euclid's narrow length-area-volume dimensional distinctions, he shoehorned *bound extensions* and *free additions* together indiscriminately like so: A bound line segment and a free bundle of two points he called a *line*, indiscriminately. A bound patch of plane and a free bundle of two line segments he called a *plane*, indiscriminately. And so on. See for yourself:

There are seven types of spatial magnitude, divided into four orders:

1st order	1. Simple or multiple points
	2. Straight lines of definite length and direction
2nd order	3. Definite parts of definite infinite straight lines
	4. Plane areas of definite magnitude and direction
3rd order	5. Definite parts of definite infinite planes
	6. Definite volumes
4th order	7. Definite volumes [p289]

Grassmann's way of saying ***bound*** is to say *definite parts*; so he is here describing points and free vectors, dimensions **1** and 1;[17] bound vectors and free bivectors, **2** and 2; bound bivectors and free trivectors, **3** and 3; and bound trivectors, **4**.

His dimensional shoehorning had two infelicitous consequences. First, it left informal the rule-induced distinction between free and bound (read 6 and 7). Second, it gave *ambiguous* dimensions—literally *two-valued*—to his order formalities, *which therefor could not be interpreted as dimensions* (tho they really were): Grassmann's *order* was for him merely an abstract formality needed to oil the gears in his algebra. This confusion has not yet been untangled by our mathematical community; so we have still not acquired the extraordinary expressive power of Grassmann's full algebra, as enhanced by Clifford.

Which brings us to the three final heroes in this story: In 1876 **William Clifford** unified Grassmann's inner and outer products (without actually saying so) into a fully informative *geometric product*, valid in the free sublanguage of Geometric Algebra.[18] It lay fallow until **David Hestenes** began to further enhance and promote it beginning in the 1960s.[19] Over the next half century he developed it into a very expressive and successful algebra. But a purely free one, de facto: the bound foundation for Grassmann's algebra had by then been lost (in plain sight). Only one contemporary mathematician, to my knowledge, has intimate understanding of it: **John Browne** who recently enhanced it into an elegant Projective Geometry, able to *directly articulate* what had been either largely discussed before, or else indirectly articulated using ad hoc models.[20] Few are aware of Browne's algebra because few have begun to articulate points like he has.

Let us finish by reviewing the very *reasonable effectiveness*[21] of that simple idea, for which the imaginary number can serve as a poster child. Mathematicians are well acquainted with the delights it engenders: Fundamental Theorem of Algebra, contour integration, de Moivre's Formula, Euler's Identity, Mandelbrot fractals, Julia fractals, Cauchy–Riemann equations, holomorphic functions, on and on. A person could spend a happy lifetime exploring these intricacies, and some do in fact.

All of this intricacy arises even with the imaginary number still trapped within the field axioms. It got stuck in there because it fortuitously obeys the rules real numbers obey. This narrow obsession with syntax is needlessly confining.

By shifting your gaze to semantics, to *formal geometric dimension* in particular, imaginary i becomes unimaginably more delightful. It makes a hasty departure from the dimension-deprived field axioms, and at last enters its native land: dimension-rich Geometric Algebra—an Oz-shift from black-and-white to blooming color.

The purely free sub-algebra begins to expose the intricacy. It shows not only that i has dimension, but even better it shows that i is *composite*. A *bivector imaginary*, for example, arises from extension of *two* free vectors, a fact that allows it to articulate orthogonal turns within its plane. With i stuck in the field axioms, it had seemed as boringly un-dimensioned and non-composite as a real number. With i literally *freed* in its native land, its lovely composition comes into view at last.

Lovely *compositions*, I should say, *plural*: bivector imaginaries of course come in endlessly many sizes and orientations; but that is just the start of imaginary plurality: there are also *trivector imaginaries*, and in fact endlessly many higher-dimensioned ones. These intricacies further multiply themselves like so: variously dimensioned, variously oriented imaginaries can roam around within their domains to collaborate with each other. For a trivial example, three independent free bivectors in physical space can collaborate with a scalar, thereby generating the rotation-articulating so-called *quaternion algebra*, finally exposed—a century and a half after it was devised—for what it really is.

Even such magnificent intricacy gives only a peek at imaginaries' rich composition. When they descend into—or rather *ascend out of*—the *full* Geometric Algebra, their true composition comes into view at last: *They ultimately arise from neg-commuting points*. A free bivector, for example, is an intrinsically composite sum of two bound vectors, as you just discovered. Each of those bound vectors, in turn, arises from neg-commuting extension of two points. So…

…all things with geometric locus are ultimately composed from points.

This is a timeworn idea, endlessly *discussed*, but not fully *articulated* until Grassmann began doing so. When you finally start doing so yourself, imaginaries become trivially imaginable. When most of us start doing so, we will at last emerge into the much more fertile *demystified* phase of mathematics, ushered in, may we never forget…

…by simply articulating points.

Endnotes

[1] Grassmann, Hermann, *A New Branch of Mathematics, The Ausdenungslehre of 1844, and Other Works*. p15. Translated by Lloyd Kannenberg, Open Court, 1995. My emphasis. The renowned Gauss read these words after receiving a gratuitous copy of Grassmann's book from him. He replied that he had already explained the idea of geometric imaginaries in 1831; and he did not have time to read the rest of the book. In fact he had merely described a *geometric* ***model*** of imaginaries and scalars like the Argand model we use today, not *geometric* ***numbers*** like Grassmann had achieved. The weighty authority of Gauss's dismissal and his presumption of priority discouraged Grassmann from pursing his own nascent *imaginary* ideas further.

[2] *Euclid's Elements*, p99, translated by Thomas Heath, Green Lion Press, 2010.

[3] Grassmann pursued simple creativity rather than intricate cleverness. For example, his book on *Arithmetic* inspired the so-called *Peano Axioms* that characterize integers inductively, as Peano acknowledged. Peano was an enthusiast of simple creativity, and he said so when promoting the "*fecundity and simplicity*" of Grassmann's extension algebra in his 1888 book, *Geometric Calculus*. Ironically, his meticulous polishing of that algebra attracted the attention of only a few curious eccentrics, whereas his relatively trivial polishing of Grassmann's arithmetic attracted the attention of the entire mathematical community (which celebrated him and ignored Grassmann).

[4] (Euclid's) *Elements of Geometry*, p8, translated and *extensively* elaborated by John Playfair, J. B. Lippincott, 1857.

[5] *Independent* in the usual technical sense: no point can be derived from scaled addition of the other points.

[6] *The Rules of Algebra (Ars Magna)*, Girolamo Cardano, translated by Richard Witmer, MIT, 1968. In Chapter I, Cardano dwells on "*true*" and "*false*" solutions, meaning positive and negative ones. He shows that false solutions often profitably lead to true ones. Finally, in Chapter XXXVII he gets around to "*The second species of negative assumption [which] involves the square root of a negative*." He shows that "*putting aside the mental tortures involved*", it can also lead to a true solution that "*truly is sophisticated*". He then concludes: "*So progresses arithmetic subtlety the end of which, as it is said, is as refined as it is useless*." p220.

[7] Pragmatic mysticism is often fruitful: Newton employed it in developing a law of gravity, whose physical cause mystified him, as he frankly admitted. On the other hand, rejection of mysticism can also be fruitful: Maxwell rejected mystical Newton-like *action-at-a-distance* theories of electricity in favor of one propagating thru a *physical ether*. His theory was successful but his motivation for it was subsequently replaced by an *abstract ether*, a *field*. Fields constitute a fruitful fallback to a reduced kind of mysticism, an enhanced kind of pragmatism.

[8] Mahoney, Michael, *The Mathematical Career of Pierre de Fermat Fermat*, p28. In subtle contrast, Richard Witmer translates this in Viete's *The Analytic Art*, p11 as "*assuming that which is sought as if it were admitted [and working] through the consequences [of that assumption] to what is admittedly true*".

[9] ibid, p42: "*Although Fermat paid lip service to Viete's Law of Homogeneity, his use of algebra in geometric situations clearly shows that he no longer attached dimension to the degree of an expression*."
p71: "*Fermat for his part could not appreciate that the tradition of Viete had run its course, that in its efforts to restore the mathematics of the ancients it had in fact created something entirely new, which demanded that mathematicians stop looking toward the past and fix their eyes on the open-ended future. To have been a 'second Viete' was at once Fermat's glory and his tragedy*."

[10] Descartes, *The Geometry*, 1637, p2, translated from the French and Latin by David Eugene Smith and Marcia L. Latham, Open Court, 1925. Dover reprint 2010.

[11] Descartes asserted, in his *Rules for the Direction of the Mind*, that "*I myself was long deceived by those names*."

[12] Grassmann, ibid, p285.

[13] The whole meandering story is told in *Grassmann's Upbringing of his Fertile Mind*, found in the Geometric Algebra page at gary-harper.com/

[14] The traditional terms are *antisymmetric, skew symmetric* or *anticommuting*. *Antisymmetric* and *skew symmetric* are vague, and they improperly specialize *symmetric*; *anticommuting* is merely vague. All three terms are needlessly pompous and require parsing. *Neg-commuting* requires no parsing at all and is, dare I say, humble and precise.

[15] Whereas *free* elements have *separation*, ***bound*** elements have ***magnitude***. *Separation* is the seminal metric; *magnitude* derives from it as separation of the *free part*. See the *Semantic Formalities?* chapter of *Playing With Geometric Algebra*, available at at gary-harper.com/

[16] Here are some hints: slide your resulting bound vectors along their confining lines to conjoin them pairwise at their tails. That will give each pair a common extension factor. You can hand it to the distributive law in reverse, thereby collecting four terms into two separate but exactly opposite bound vectors. Sketches will be crucial. Formal freedom can be exploited to simplify, *considerably*.
For maximum enlightenment ponder these questions: How many different ways can such pairwise conjunction be done in a plane? In a volume? Can *three* of the terms be combined separate-but-exactly-opposite to the remaining term? The ultimate result of such multiple bound-vector addition might seem to be a single bound vector. If so, what is its magnitude? Where is it?

[17] Yes, points and free vectors have the same *order* (now called *grade*) meaning the ***same numeric dimension***, namely one, neglecting composition; ***which must not be neglected*** if you want to distinguish *free* from ***bound***, *separation* from ***magnitude***. That is why the ***bold non-composite*** versus the *non-bold composite* distinction is so crucial.

[18] Clifford's seminal paper on this is *On the classification of geometric algebras*, found in his *Mathematical Papers*, p397. Also relevant are *On the hypotheses which lie at the bases of geometry*, p55; *Preliminary sketch of biquaternions*, p181; *Applications of Grassmann's extensive algebra*, p266. These papers were all written in the 1870s. Clifford-the-geometer, like Peano-the-polisher, was a rare enthusiast of Grassmann, unlike Hilbert-the-mathematician and most others, who considered him bafflingly innovative but woefully non-rigorous.

[19] Hestenes's *New Foundations for Classical Mechanics* is the book that first got me fascinated with Geometric Algebra. It is a textbook replete with many figures, exercises and good geometric intuition; one of my all-time favorite books. Hestenes, however, considers his pithy *Clifford Algebra to Geometric Calculus* to be his true magnum opus. It began life as a sequence of scholarly articles. As with most such articles, if you want to mine it bring your own figures, your own geometric intuition, and plenty of dogged determination.

[20] Browne, John, *Grassmann Algebra*, Barnard Publishing, 2012. When extension exceeds the ceiling of its domain, the result vanishes, which discards useful information. Such information can be retained, Grassmann realized, by wrapping back around thru the scalar floor with a dual product he called the *regressive product*. Unfortunately, this wrap-around gave his floor and ceiling ambiguous dimensions.

Browne removed that ambiguity using dual axioms that *directly articulate* Projective Geometry in the simplest possible way. For example, the well-known duality between a point and a line in a plane becomes a mere algebraic expression; and that expressiveness automatically extends to higher dimensions. This is the *non-metric* part of his algebra.

The metric part articulates the inner product, and it seems to be under development. In its present form it can, I believe, generate contradictions unless it is wielded carefully; but that is beyond the scope of this paper (and of course I may be wrong). It will be examined in the *Synthesis and Style* chapter of *Playing With Geometric Algebra*, cited previously.

[21] Nobel laureate Eugene Wigner celebrated what he considered *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, in *Communications in Pure and Applied Mathematics*, vol. 13, No. I (February 1960) He dwelled on the intrinsic mystery of complex numbers and their inexplicable necessity in Quantum Theory.

Hestenes adamantly opposes these ideas, showing not only that complex numbers lose their mystery in Geometric Algebra, but also that their use in Quantum Theory is quite explicable: rotations of some kind typically. And that is merely his perspective from Clifford's purely free sub-algebra. When physicists begin using Grassmann's *full* algebra, Clifford-enhanced, I am betting complex quantum mysteries will evaporate, leaving us with the entirely *Reasonable Effectiveness of Mathematics*. Good intellectual tools convert unreasonable machinations into reasonable calculations.