Grassmann's Upbringing of his Fertile Mind

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Hermann Gunther Grassmann's mind was perennially wondering, exploring, creating; and he persisted in its upbringing until it lay dying in the incongruous body of an old man. That old man knew much about persistent upbringing: he had raised nine of his eleven children; he had taught high school full time his entire adult life; he knew how to change minds.

So it is not surprising that even in his last year of life, 1877, he was still changing his own mind. In that year, as he was preparing a second edition of his first book, his 1844 *Lineal Extension Theory*,[1] which seemed to him "certain to be more pleasing to philosophically inclined readers"$_{\text{p21}}$, he added several footnotes describing previous concepts as "sterile". Exactly when they became sterile to him is not certain, but it is certain that the process was a patient one because "…the time for my research was extremely paltry and measured out piecemeal by virtue of my official position."$_{\text{p17}}$

Regardless of when he changed his various minds, it is our exploratory good fortune that he left many of his preconceptions intact in his first book even when he realized, after writing them, that they were wrong. (You shall see several examples, some startling.) Like all curious new authors, Grassmann learned as he wrote; but unlike most he did not revise much, especially the book that had clearly taught him most, his first *more pleasing* one, whose methods "probe more deeply into the essence of the subject…".[2] Even for its second and final edition he "left unchanged the text of the first"$_{\text{p21}}$ despite having explicitly discarded some of its core concepts and symbolism in his mature "so dissimilar" 1862 *Extension Theory*. He simply augmented his seminal 1844 text with footnote annotations, three expository appendices, and an index of defined terms.

So a reader can pore over his augmented text and observe the evolving trajectory of a young curious creativity. This does require some time (I have been at it leisurely for twenty some years) for a baffling reason:

His nascent algebra began with intuitive operations in physical space, but he soon realized that their *generative nature* actually transcends that space (quite unlike Hamilton's unintuitive operations). So you might expect that Grassmann would have provided his generative intuitions to his readers so they too could retrace his path beyond physical space. That transcendence, in fact, motivated him to do just the opposite: he intentionally disengaged his operations from their intuitive foundations and instead presented them as lofty abstractions. Fortunately he did deign to illustrate his abstractions with a few *Applications* to mere geometry.

So you can focus on his *Applications* to rassle those abstractions into some kind of concrete geometric clarity. However, you can't be sure he won't later consider them sterile;

and even if he doesn't, you still can't be sure they really aren't sterile, given the *paltry time* available for their development. The big payoff for grappling with such perplexity is the exceptionally expressive geometric language you will extract.

Extracting Grassmann's algebra is further complicated by his having accidentally begun at the *geometrically free end* rather than the *geometrically **bound** beginning*. This naturally generated misconceptions, some of which he clearly realized and eventually revised; some he vaguely realized and slightly revised, or not; some he did not realize at all because time just ran out. It would ill behoove us to approach such confusion with anything but sympathy—we too have accidentally begun backward at the free end, as will shortly become clear. The difference is that we have not yet ended at the bound beginning as Grassmann eventually did; so he has much to teach us.

To prepare for his lofty lessons, we shall first preview them from a comfortable distance, and then take a glance at a forward path thru them. So forearmed, we shall finally undertake a brisk step-by-step tour of his actual backward path. It was so creative and meandering that it left most readers bewildered; but your fresh foreknowledge, I am hoping, will leave you enlightened instead, if still bemused. (I am still bemused, lo these twenty-some years.)

Grassmann's gestalt

Grassmann's seminal idea, his gem, is the idea of extension: extend a point, α say, to point β to produce a directed line segment $[\alpha\beta]$. That line segment can itself be extended sideways to produce a directed parallelogram patch of plane. *That* can then be extended sideways to produce a directed parallelepiped chunk of volume. Grassmann stopped there, briefly, because he had run out of space.

But why stop there? One can keep going, at least *abstractly*. With that thought Grassmann then proceeded beyond physical space; with that thought he acquired his enthusiasm for abstraction.[3] His evolving algebra subsequently lost its original grounding in geometric meaning. Absent that discipline, it acquired inconsistencies we shall have to remove to understand and use it.

Extension was the one original idea that began well grounded, and it proved valid even in the abstract realms Grassmann began to explore. He naturally expected his fresh new extensions to interact with each other, which seemed to require them to conjoin by translation. That is what caused him to accidentally begin at the end, *purely free*, like so:

He initially convinced himself, via a bogus philosophical argument (scrutinized later), that his line segments (and consequently all higher extensions) are free to move sideways parallel to themselves. *Such freedom is inconsistent with line segments generated by point extension*, which *the algebra* binds to the lines thru themselves,[4] philosophy notwithstanding. At the beginning, however, Grassmann did not have such *formal* point extension; and, at the beginning, he believed that translation freedom was necessary for his addition; so he conjured a "theorem" to make it true.

Fortunately, his translation argument *is* true for a different kind of vector, *an un-extended one*. His argument had serendipitously articulated such vectors, even tho *he had not yet discovered their proper formal representation*.

So he began with a purely free algebra, misinterpreted, that he called *extensive magnitudes*. That infelicitous terminology seriously misleads about the *intrinsically unextended* nature of his primitive free elements, which Grassmann called *displacements*, meaning *free vectors*. Ironically, his preliminary and oft-repeated assertions that his displacements *were* extended line segments, *roving ones*, motivated him to extend them, *as primitive entities*, along themselves. That fortuitously generated the proper roving formalities, but they were interpreted in a way inconsistent with his eventual bound algebra.

In short, he began with a *purely free* algebra generated by unwittingly freeing *bound* line segments, as you shall see in detail.

A free algebra arises spontaneously from a bound one, but not the other way around. Nevertheless Grassmann was in command here—not the algebra he was creating—and he eventually managed to go the other way, from free to bound, from end to beginning. He annotated his journey with algebra, but its presumptions often made it philosophy.

Even tho his journey was algebraically backward, and even tho some of its assumptions were contradictory, it was nonetheless an astonishing act of creativity that puts us moderns to shame: We too have begun at the end, geometrically free; but not one of us has ever independently made a journey backward from free to bound. Indeed, very few of us have even managed to decipher Grassmann's account of his backward journey, altho we have had centuries to do so. We are still stuck in the purely free algebra, blissfully unaware of the more expressive bound one that can generate it. So Grassmann's fortuitous reverse engineering merits scrutiny.

Here is how it all started: After establishing in the first half of his book the simple laws of his free algebra and some of their intricate consequences, Grassmann posed himself a kind of mathematical analogy: find a vector whose sum with another vector equals the sum of two given vectors—find a sum analogous to a given sum.[5]

He became intrigued that his solution depended only on the vector headpoints, and not on their common origin, which could be anywhere. That led to the idea of dispensing with the origin and coalescing a pincushion cloud of vector headpoints to a single point. Grassmann's careful analysis required such a maximally condensed point to have what he apologetically called *weight*, some non-zero number.[6] These newly discovered, *algebraically disciplined* points became the primitives for his subsequent bound algebra. He called them *elementary magnitudes*.

Yes, *elementary*—Grassmann has finally ended at the beginning, halfway thru his first book.

He realized his new weighted points were *elementary* because, even tho they were obviously *bound*, they unexpectedly generated his entire *free* algebra, his so-called "*extensive magnitudes*" that had occupied the first half of his book. Here is his epiphany: there are clouds of headpoints that cannot coalesce to a single point, and they are all *weightless*. Some of these weightless clouds, as you might expect, just disappear. *But most don't*. A persisting *zero-weight* cloud can be maximally condensed to a subtraction of two separate unit points, $\beta - \alpha$ say.

Best of all, the algebra does not care where this two-point bundle resides—it can move anywhere parallel to itself.[7] *Grassmann has just discovered the proper formal representation for his displacements*.

He realized he had done so because he had seen such geometric freedom before; he had dwelled on it in fact—he had painstakingly established it as a theorem: roving β–α *must be extension* $[\alpha\beta]$! (*No it must not: subtraction does not fill-in between points—**increment dimension**—like extension does*.) So he wrote, in the middle of his book, the following equation, neatly centered for emphasis:

$$[\alpha\beta] = \beta - \alpha$$

This can properly be called Grassmann's unanticipated and premature *central equation*. Notice the transposition of symbols from one side to the other. This arises because a point subtraction expresses its tail at the right whereas Grassmann's bracketed *extension-to* expresses its tail at the left. This minor kink in his meandering journey gives evidence for just how unanticipated this equation was. Had he known it when he began writing, he likely would have expressed point extension and point subtraction in the same order, for consistency, as Hamilton did in a different context.

This central equation may look like mathematics, but it was really philosophy. Altho it finally introduced Grassmann to the bound foundations for his free algebra, it was otherwise a calamity for him and for us: From left to right it confuses bound with free, multiplication with addition, prose with mathematics. Perhaps worst of all, it confuses dimensions: an extension of points, *when properly formalized*, acquires Grassmann's *order* 2 (as he eventually stated—see below), whereas a subtraction of them retains *order* 1 (as he always stated). This central equation, judging from historical evidence, confused nearly all of Grassmann's readers. It sure confused me for many years.

Unravelling this 'equation' will finally free us from a horde of confusion. We shall unravel it by peering over Grassmann's shoulder as he travels backward geometrically. So, to gain perspective on where we are going to eventually wind up, it will be essential to first take a quick trip forward. This will be a synopsis of my *Synopsis*,[8] so you may go there for more detail.

Geometrically forward, with formal extension

To consistently tie the geometry to the algebra, derive everything from three primitive ideas: 1) the idea of geometric points having fixed distances among themselves; 2) what it means to summarize points, namely, order doesn't matter, grouping doesn't matter, and a point summarized with *Nothing* is just the point itself; and finally, Grassmann's eventual gem; 3) extending something to a point just sweeps it directly there, filling in as it goes, which increments dimension.

This is *eventual* because the ad hoc symbolism in his first book, and his abstract interpretation of it, initially prevented him from perceiving it clearly. By the time he had written his second book he had perceived it clearly, as you shall see.

The three seminal ideas establish precise symbolism that immediately generates a free vector, β–α, from fixed points α and β. Its freedom arises because the sum of β–α vanishes to infinity, leaving only the summand bundle for the algebra to work with. The algebra is of course happy to work with either a sum or its summand bundle—they are literally equal according to the axioms. However, if the sum is not available, *as it is not* for β–α, then the algebra *must* work with the summand bundle.

What is surprising is that the algebra allows this ostensibly bound bundle to move anywhere parallel to itself, which was Grassmann's epiphany. If this seems mysterious, as it likely will since our mathematical community has not acquired it yet (or rather *almost* acquired it a century ago, but let it slip away), work thru the details in my *Synopsis*. They appeal directly to geometric intuition. If you prefer abstract algebraic reasoning, work thru Grassmann's point-cloud derivation previously sketched.[9]

You can use free bundle β–α to move the endpoints of one of Grassmann's *actual* line-segment extensions, [γδ] say. Just place the bundle's negative point (its tail) over point γ and then add, and similarly for δ. This will annihilate each of those points, leaving as residue new positive points γ′ and δ′. You can then extend these points to generate a new line segment, [γ′δ′].

By construction this is a parallel-translated version of [γδ], which it ***will almost never equal****, contrary to Grassmann's preliminary very detailed assertions* (coming up). To be equal to [γδ], it would, for starters, obviously have to be expressible entirely in terms of points γ and δ. This is generally not possible because translator β–α is generally not so-expressible.

Well then, suppose the translator were so-expressible. Then it would be a scaled version of δ–γ, *which translates* [γδ] *somewhere along the line thru itself*. In this case extension just ignores the translation, thereby making the translated version of [γδ] equal to it. In other words the algebra of extension ***binds*** an *actual* extension [γδ] to the line thru itself.

Of course such binding happens only after point extension has been properly formalized, which it was not in the first half of Grassmann's book. In fact points are not even algebraically expressible there, pervasive point notation notwithstanding. Such notation was not really part of the algebra—it was merely a crutch for geometric reasoning. Grassmann was able to properly formalize point extension only after his book had finally taught him about weighted points, about *formal* "*elements*". Until then he simply had no way to understand the formal distinction between geometric bondage and freedom.

To begin understanding that distinction yourself, and to begin seeing what you shall have seen, you now have sufficient perspective to turn from this preparatory side trip geometrically *forward*, with *forma*l extension, to examine how Grassmann actually meandered…

Geometrically *backward*, with *informal* extension

After launching his book with a disquisition dispatching the *Derivation of the Concept of Pure Mathematics*, Grassmann then turned his philosophical gaze on the *Derivation of the Concept of Extension Theory* beginning on page 25. That section begins like this:

> Each particular existent brought to be by thought can come about in one of two ways, either through a simple act of *generation* or through a twofold act of *placement and conjunction*. That

arising in the first way is the *continuous form*, or *magnitude* in the narrow sense, while that arising in the second way is the *discrete* or *conjunctive form*. (His definitional italics)

As you scan forward trying to get some meaningful glimmer of the topic at hand, *Extension Theory*, you learn that "The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete." That tends to dissolve any distinctions you may have extracted. Slogging forward reveals that "It scarcely needs further demonstration that the concept of number is hereby completely exhausted and precisely delimited, and likewise that of combination." Scanning back looking for previous demonstrations of number or combination that you must have missed reveals that there aren't any, if by "*demonstration*" Grassmann meant *concrete example*. Marching ahead another page, you finally encounter *extension*, and learn that "The intensive magnitude is thus that arising through the generation of equals, the extensive magnitude, or *extension* that arising through the generation of the different." Trying to clarify this, you learn a half a page later, that "It is thus somewhat as if the intensive magnitude is number become fluid, the extensive magnitude combination become fluid." Just as you begin to despair of ever encountering anything concrete to grasp with your mind, Grassmann suddenly reveals, on page 27, the geometric fount from which his abstractions had flowed: "The best example we can offer for the extensive magnitude is the line segment (displacement)."

Eager to fill out this remarkable insight, you keep reading, but your guide wants to first indicate the kind of trip you will be taking, the form of things you will encounter along the way, and the kind of mathematical gear you should take along.

After outfitting yourself for the trip you finally set forth on page 46 by learning how extension generates line segments: "Here it is a generating point that assumes a continuous sequence of positions; and the collection of points into which the generating point is transformed with this evolution forms the line." Ah ha—a line segment generated by sweeping one point to another! Two pages later, you learn how to symbolize it: "We tentatively represent the displacement from the initial element α to the final element β as $[\alpha\beta]$." In 48 *extensive* pages you have finally nailed down extension's essence:

Sweep point α to point β to form directed line segment $[\alpha\beta]$.

To this notation, Grassmann provided, in his last year of life, a long appendage to a footnote that began by reasserting that "This notation is only tentative…"[10] That is too mild. The footnote should have begun:

Achtung! This notation, when applied to a displacement, seriously misleads about its *intrinsically unextended* nature. Moreover, this notation is not yet disciplined by the rules of algebra even tho it often appears to be. Consequently, it may confuse you. For perhaps your entire life.

To his credit, Grassmann did eventually formalize this notation in his 1862 *Extension Theory*; and it subsequently challenged his interpretations. His 1877 appendage to the footnote reveals his response to the challenge. It is perhaps the clearest paragraph Grassmann ever wrote. We shall examine it in its entirety at the end of this paper, when you have some hope of understanding its implications. That will require first examining the intellectual havoc a pre-formal notation like $[\alpha\beta]$ can wreak.

You don't have far to go to get started. After having mis-generated displacements *by extension*, Grassmann begins *Addition and Subtraction of Similar Displacements* (having parallel directions) on page 48, and then *Addition and Subtraction of Dissimilar Displacements* (having non-parallel directions) on page 51. He is extensively careful about this, showing how contiguous higher-order extensions (planes, volumes, etc.) induce a corresponding sum of primitive-order displacements. However, this is not careful mathematics; it is careful philosophy. The philosophy fortuitously matches the mathematics so long as he stays contiguous. But he is determined not to stay contiguous because he wants to conjoin arbitrary displacements.

So, on page 58 he *informally* adds a displacement to each endpoint of another displacement (as I had done formally with a line segment in the previous section) and then he claims that the parallel-translated result equals the original. Here is his reasoning:

Thus if $[\alpha\beta]$ is the original displacement, and if $[\alpha\alpha'] = [\beta\beta']$, then

$$[\alpha'\beta'] = [\alpha'\alpha] + [\alpha\beta] + [\beta\beta'] = [\alpha\beta]$$

since $[\alpha'\alpha]$ and $[\beta\beta']$ are removed by the addition as opposed magnitudes.

But wait, aren't $[\alpha\alpha']$ and $[\beta\beta']$ *already* parallel-translated displacements *that are equal*? A quick look back at Grassmann's figure on page 52 reveals that they are because the former is conjoined to the tail of $[\alpha\beta]$, the latter is conjoined to the head some distance away; yet *he presumes they are equal* ("and if $[\alpha\alpha'] = [\beta\beta']$…"). So Grassmann is assuming that *some* parallel-translated vectors are equal, namely $[\alpha\alpha']$ and $[\beta\beta']$, which makes *other* parallel-translated vectors equal, namely $[\alpha\beta]$ and $[\alpha'\beta']$.

Hence, he could have said validly, with a little more care, 'If some sideways-translated displacements are equal, then all translated displacements are equal.' (This assertion would be vacuous for *actual* point-extension line segments $[\alpha\beta]$ and $[\alpha'\beta']$ because sideways-translated versions of them are never equal.) Here is what he actually said:

"*If all elements of a displacement are evolved by the same amount, the resulting displacement remains equal to its original*." (His italic emphasis.)

By assuming what he wanted to prove he was able to prove it, which shows how eager he was to do so. A point algebra—a *bound* algebra—would have immediately informed him of his mistake (and eventually did). It would have told him that his roving argument *is not true* for a point-*extension* "displacement" $[\alpha\beta]$, but *is true* for a point-*subtraction* displacement, $\beta - \alpha$.

That *algebraically* roving displacement arose only after he had acquired the formal points needed to articulate it. After acquiring such points, his philosophical demonstration of geometric freedom became superfluous—the algebra itself automatically induces such freedom.

Apparently Grassmann did not have time to revise his early conceptions, else he would have extensively revised the 1877 edition of his 1844 book to better align it with the mature conceptions in his 1862 book. He seems to have been motivated to let bygones be bygones and march onward. That tactic did serve to maximize his creativity, if not his rigor.

For the purpose of unrigorous creativity, his fresh new points were extraordinarily fertile. They gave him the abstract "*elements*" for a bound algebra that he developed in the second half of his book, right after his central equation. That equation is worth another look for the perplexing comment that follows it:

$$[\alpha\beta] = \beta - \alpha$$

> These two expressions therefore only represent different notations, and since the former is arbitrary, the latter necessary, we prefer to drop the former… p161

That comment is perplexing because his "*necessary*" subtraction notation is used for only one page! The page after reverts to his "*arbitrary*" bracketed extension notation. In fact, *the entire rest of his book uses bracket notation* for displacements, except in those few places where he wants to exploit endpoints. Even then he typically entangles subtraction notation with bracket notation.

Why did he cling so tenaciously to bracketed extension for free vectors? I believe it was because *he never abandoned his seminal idea that displacements are line segments*. *Never*, as you shall see at the end, at his end. Line segments arise from point extension, not point subtraction; so he kept using extension notation for them in his first book, *even after he had discovered that it contradicts his roving formalities*. Clearly, his geometric interpretations drove his abstract symbolism, despite his earnest efforts to have it the other way around.

The price he had to pay for his misinterpretation was high. This becomes painfully clear in the very next section, *The Extent to Which the* ***Displacement*** *can be Interpreted as a Product*, on page 169 (my boldness). Only eight pages prior he had boldly declared that displacement [αβ] is actually the *sum* β–α. And now he is arguing that this *sum* is a *product*?!

Not really, because his "*necessary*" subtraction notation is nowhere to be found within his reasoning. Instead he proceeds, as usual, with bracketed extension [αϱ] which is indeed a product, ***but not a displacement***. He shows that it scales and distributes like a product does, and then concludes that… "Further, since [ϱϱ] is zero and [ϱα] = –[αϱ], it follows that this is an outer multiplication."

Could Grassmann possibly have asserted that a ***displacement*** *"**is** an outer multiplication*" if he had dropped his *arbitrary* [αϱ] notation, as he had promised, and instead used his *necessary* ϱ–α notation? If he had written, 'Further, since ϱ–ϱ is zero and ϱ–α = –(α–ϱ)', he might have concluded that 'it is worthy of remark that subtraction of points commutes negatively like outer multiplication of them would.'

But it is also worthy of remark that *subtraction* does not distribute over addition like *outer multiplication* would; and it is especially worthy of remark that *subtraction* does not increment dimension like *outer multiplication* would. Here you see the problem with duplicitous notation: it engenders duplicitous concepts. Grassmann's remarks become more understandable when you realize that they constitute his *pre-formal* introduction to actual point extension, which is indeed "*an outer multiplication*".

At this point a reader begins to understand that Grassmann's first book, owing to the *paltry time* available for its writing, is a public beta version of his fertile ideas—he is kindly giving us his initial raw creativity, *left in its formative state*, not yet well disciplined.[11] I, among many others, deeply appreciate his generous creativity, premature tho it may have been. Would that more mathematicians had the courage to boldly blunder where others have never blundered before. Worthy efforts can always be tidied up; unworthy ones can be declared sterile ex post facto.

To remove any doubt that the early Grassmann considered all extension, *even his seminal free vector displacement* [αβ], later known as β–α, to be a product, go back to page 76 where he writes,

> Further development thus demands the generation of a new species of extensions. The nature of this generation follows at once by analogy with the way the extension of first order ["**displacement**"] was generated from the element, since in the same way one can again subject the collection of elements of a **displacement** to another generation. (My boldness again.)

Which is to say, just as one extends a point sideways to generate an informal line segment [αβ], formalized later as β–α, one can extend that line segment sideways to generate a parallelogram patch of plane, and so on up. Then he shows on page 78 that such *extension obeys the distributive laws of multiplication* so "…our conjunction is now established as multiplication, and thus we immediately introduce the multiplicative symbol [the dot .] for it."

However, Grassmann's multiplicative dot most certainly did ***not*** extend "***in the same way***" as his bracketed "extensions of first order" had done. It could not have done so because he mistakenly freed those extensions, so they had no moorage from which to be extended to a point *in the same way* as he had generated them. He had no choice but to extend his roving line segments ***in a different way***, namely *along other roving line segments*, rather than to a point. Such freed line segments had to be treated as *primitive wholes* whose bracketed endpoints could not interact formally with other extensions. (How could they?—that would have freed his points too!)

This new *along*-extension therefor required a different notation than his original *to*-extension, a dot . rather than brackets [ ]. This dot was a genuine product; but actually a product of *formal point subtractions* like β–α, not a product of *informal point products* like [αβ]. Fortunately, point products are finally about to become formal for Grassmann.

They begin on page 171 with the *Outer Product of Elementary Magnitudes Formally Defined*. There is immediate fog, a cloud of points. This is a peculiarity of Grassmann's fondness for abstraction, that he begins in the most general way he possibly can with "a product of *n* elementary magnitudes of first order" (an extension of *n* points). Most readers might prefer beginning with an extension of just two points, and working up from there.

Such readers can take hope in the following unusual remark: "Our problem thus remains in particular to give our concepts the greatest clarity, and to illustrate their concrete representation." Here is the first great clarity:

> A rigid elementary magnitude of *n*-th order can be represented as a product of an element with an extension of (*n*–1)-th order, and this extension, which we call the *divergence* of that elementary magnitude, is completely defined by it, but as its element any one belonging to the system defined by the elementary factors of the elementary magnitude may be adopted.

Clarity or not, this is actually the *fundamental theorem* of the full Geometric Algebra. It describes the very elegant relation between free and bound. It is truly the proper algebraic link between the first *free* half of Grassmann's book and the second *bound* half. If only

Grassmann had used language to *communicate*, rather than to exalt his innovative ideas, he might have written this:

> *A **bound extension** is its unique **free part** extended to some **unit point** within its **confining space**. Such extension, like all point extension, increments the dimension of the free part.*[12]

Grassmann's bad language was as disastrous for him as Mr. Bligh's bad language had been on *HMS Bounty*. He couldn't bring himself to call an "element" a *unit point*, a "rigid elementary magnitude" a *bound extension*, a "system defined by the elementary factors of the elementary magnitude" its *confining space*, a "divergence" a *free part*. Consequently he confused everyone, himself perhaps less than others.

Possibly the worst language in his fundamental theorem is his use of *rigid elementary magnitude* for a bound extension, and his use of *divergence* for its free part. *Rigid* is contradicted by the algebra, and it caused Grassmann (and Peano)[13] to seriously misinterpret bound extensions as *vertex structures*. *Divergence* does not properly indicate its free geometric contradistinction with bound—with so-called *elementary magnitudes*. Grassmann abandoned these terms and many of the ideas associated with them in his 1862 *Extension Theory*, to which we shall shortly meander. To get there, we first hike thru *The Vertex Structure* on page 174, which begins like this:

> According to the concept established in the preceding paragraph, the product of two elements, α, β is the displacement $\alpha\beta$ bound to, and thus as it were rigidified by, the elementary system defined by α and β.

Here you see, *for the very first time*, Grassmann ***formally*** extending point α to point β. Here you see him recognizing for the very first time that such an extension is *not* free to move parallel to itself—it is "*bound*" to the line thru itself ("the elementary system defined by α and β"). Grassmann's statement merits close scrutiny because it contains an important subtle inconsistency and a relatively trivial bold contradiction.

The subtle inconsistency is Grassmann's "displacement $\alpha\beta$". Didn't he mean "displacement $[\alpha\beta]$"? Aren't brackets his de facto notation for displacements? Yes they are. Hasn't he now happened on the proper way to formalize extension $[\alpha\beta]$? Yes he has, and such an extension is a *bound* vector, as he has just discovered.

But he has a conundrum: he can't use his bracket extension notation formally because he had informally declared that it equals displacement $\beta–\alpha$, which is an *actual displacement*, a *free vector*, as he had discovered thirteen pages previously. So he has to use simple juxtaposed point multiplication $\alpha\beta$ to formally express his bound vector. But that is *not* actually a "displacement" as he had said—it is a *displacement that has been tacitly bound*. His new fundamental theorem could have told him how that binding works:

> *Bound vector $\alpha\beta$ is free vector $\beta–\alpha$ extended to some unit point on the line thru $\alpha\beta$. Such point extension increments the dimension of free vector $\beta–\alpha$ by filling it in.*

Or rather, his fundamental theorem *would* have told him that if he hadn't expressed it with such mind-numbing abstraction; and if he hadn't already misinterpreted free vectors as being filled in. They were filled in from the beginning because he had *conceptually* generated them like bound vectors, by point extension, which he had mistakenly freed so he could easily add them. He was initially able to get away with filled-in free things because their composition was simply irrelevant so long as he did not have (bound) points to relate them to.

But now he does, and he has just discovered that formal point extension actually generates filled-in "*bound*" vectors. So now he has the daunting task of carefully distinguishing them from his mistakenly filled-in *free* vectors. (This will be an awful end-of-the-book mess that Grassmann will subsequently discard; so you might want to follow his second thoughts and just skip it. On the other hand, it does display an extraordinarily creative mind marching ahead.)[14]

Before following Grassmann's reasoning, please understand that the algebra itself is quite clear about how to distinguish free from bound: Free vector β–α is ends only, with nothing in between—it is formally *un-extended*. Contrariwise, bound vector αβ, having incremented dimension, is filled in—it is formally *extended*. Grassmann was well aware that addition of weighted points typically generates other weighted points—*addition does not change dimension*.

(I would have loved to have been able to press him on this point. I believe he would have acknowledged its validity after some thought. After perhaps *considerable anguished thought*, because it would have required him to completely revise the interpretations he had engraved into all of his writing.)

Grassmann persistently implied that addition does change dimension when the summand points are separate but otherwise exactly opposite; in which case addition magically fills in between the summands. *Filling in is extension's job*, I would have said, not addition's. Then I would have pointed out that he had carefully explained in *Extensions of Higher Order as Products* on page 77 that such geometric filling-in—such *extension*—distributes over addition, and is therefor a product. Well, addition does *not* distribute over addition; addition is *not* a product; addition does *not* fill in, esteemed Professor Grassmann.

Because the assertion that addition does ***not*** fill in *is still unfamiliar,* let me reiterate: when addition is confronted with separate but otherwise exactly opposite bound elements, its sum vanishes to infinity, hence does not exist. In such a case the algebra has no choice but to work with the summand bundle. *To repeat*, this bundle is ends only—perhaps point ends, perhaps bound vector ends, perhaps bound bivector ends … —with nothing in between.

The algebra unexpectedly gives such a bundle fixed separation and fixed direction, but no particular location. Such spontaneous bound-generated geometric freedom *is still unfamiliar* because we have not yet acquired Grassmann's bound extensions, his *elementary magnitudes*. We have not yet ended at the beginning, as he has just done in such a creative but baffling way.

Of course Grassmann could not have been familiar with bound-generated freedom either when he was just encountering it for the first time in the history of our planet. But he could have tried to make his interpretations consistent with his formalities. This he did not do because he had become motivated by abstraction. Geometric interpretation, for him, now served mainly as a crutch for a low-dimensional reader to approach his high-dimensional algebra. For such a reader, he now has to distinguish his new filled-in bound things from his previous filled-in free things and "illustrate their concrete representation". He does this in a way that simultaneously recognizes algebraic distinctions, and flouts them.

Here is the essence of his free-versus-bound distinction, version one, found on page 174: **1-dimensional** *free vector* β–α, er [αβ], was generated (as he kept repeating) by sweeping tailpoint α over to headpoint β (*no it was not, that would have been a bound vector*), thereby producing the line consisting of those endpoints conjoined to the line segment lying *between* them (his *between* emphasis). **2-dimensional** *bound vector* $\alpha\beta$ is this *between* line segment stripped of its endpoints because they, having no extension themselves, can be discarded by properties of continuity. Consequently, other bound things are also distinguished by endpoint *between*-ness, which generates *vertex structures*—triangles, tetrahedrons, etc. (Here, I rassled Grassmann's prose into some kind of concrete geometric clarity. Intrepid readers might give that a try themselves.)

Here you see him at least vaguely recognizing *ends* as the algebraic essence of free things, and *filled-in between-ness* as the algebraic essence of bound things. You also see him making the dimensional distinctions required by *free ends-only* versus *bound filled-in*. Unfortunately, the abstract language he used obscured these distinctions. He didn't actually use the concrete terms *1-dimensional* and *2-dimensional*; he used the abstract terms *first order* and *second order*. And for him, strangely, they were *always* ***both*** "line segments".

His appeal to continuity was another conjuring trick to get things to interact the way he thought they should. About it I would have asked him these questions: If continuity allows us to discard the endpoints of a bound-vector line segment, why won't it allow us to also discard the endpoints of a free-vector line segment? If it will, what is the difference between them? (He might have given a cogent response. That would have startled me yet again.)

One of the magic tricks Grassmann eventually repented of was his idea of a bound element as lying *between* all of its points of extension, which generates a *rigid* vertex structure (his *between* and *rigid* italics). He abandoned vertex structures because they obscure the elegant relation between free and bound—they require a superfluous factorial, as follows:

A *triangle* has 1/2 the area of its corresponding parallelogram; a *pyramid* has 1/3 the volume of it corresponding parallelepiped; and so on up. This partially articulates the magnitude-reducing property of Grassmann's *between*-ness. His vertex structures compound it. For example, a 4-point vertex structure is a *triangular pyramid* (a tetrahedron) having *1/2 times 1/3* the volume of a corresponding parallelepiped (its volume divided by 3-factorial).

By contrast, the free part of the corresponding parallelepiped, *having exactly opposite ends, is intrinsically parallel-sided*. Hence, its volume-*separation* equals the parallelepiped's *parallel-sided* volume.

In general, the separation of the free part of a *parallel-sided* bound *n*-vector would equal that *n*-vector's magnitude; whereas such separation would be *n*-factorial times the magnitude of a corresponding vertex structure. Grassmann's premature interpretation of bound things as "*rigidified*" vertex structures required him to use a factorial to relate them to their free parts, their "*divergences*".

His vertex interpretation is inconsistent with his algebra because point extension does ***not*** generate rigid things, as he had declared. It generates successively higher dimensional things whose sub-factors are successively free to move *anywhere* within their confining spaces. Grassmann actually knew that, and said so in his fundamental theorem:

"…as its element *any one* belonging to the system…"

Any one point in a confining space produces bound extensions that are as free to change shape as free extensions are. So Grassmann's "bound to, and thus rigidified by" is the aforementioned bold algebraic contradiction: *Bound to* does ***not*** imply *rigidified by*, else its binding point would be *a particular one*, rather than *any one*.

In fact, his original concept of parallel-swept, shape-shifting extension is as geometrically consistent for the bound extensions he has just ended with, as for the free extensions he had started with. He did not need two distinct and mutually inconsistent kinds of extension, a free kind versus a bound kind. He really needed *ends-only* versus *filled-in*.

Bound vectors like αβ are filled-in. Free vectors like β–α are ends-only. Consequently, iterated extensions of free vectors are also ends-only like so: One vector endpoint fills in a free extension in one place; the other endpoint fills it in oppositely in a separate place, as mandated by Grassmann's fundamental theorem. This would have been obvious if only he had not philosophically freed things that are algebraically bound. If only he had begun at the beginning with points rather than free vectors, we might now be fluent in the full Geometric Algebra. His 1862 book gave him a chance to begin at the beginning.

En route to it, he made a substantial jog thru the muddled middle: Shortly after he had published his 1844 book, a prize was offered to whoever could best answer whether Leibniz's *Universal Characteristic* "can be recovered and further developed, or whether one similar to it can be suggested." Leibniz had described his *Characteristic* grandiloquently as a system of geometric congruences designed for "representing to the intellect everything that depends on the imagination".[p403-4]

In response to the prize offer, Grassmann eagerly wrote an extended essay whose preamble asserted that, even tho Leibniz's "sublime idea" was "prophetic" and required "a special talent of a higher intellect", nevertheless: No, Leibniz's *Characteristic* cannot be further developed—it "lags infinitely far behind its goal", but: Yes, one superior to it has already been developed, if little appreciated, namely Grassmann's own *Extension Theory*, which "actually accomplishes what he [Leibniz] regarded as the goal of…"[p316-8]

Geometric Analysis

Grassmann's essay, so titled, was submitted in early 1846. It won the prize mid-year; and not only because it was the only submission, but also because it baffled the judges, who deferred judgement to Mobius, aptly named August, former student of Gauss, and Grassmann's known intellectual peer and correspondent. He was initially as baffled as the judges, so he painstakingly translated it into his own lucid notation and transparent conceptions—he was almost certainly the first person to rassle Grassmann's ideas into some kind of concrete geometric clarity.[15] He recommended award with the reservation that Grassmann's submission "lies rather far from the currently standard procedures of mathematical investigation".[p385] Furthest from them, in Mobius's view, was Grassmann's inner product of points.

Grassmann had stated in the *Foreword* to his *Lineal Extension Theory* that his inner product would be deferred to a second volume (dealing with *Angular Extension Theory*) which never materialized; so his development of it in this essay stands in for that. He began it applied to two parallel directed line segments, which is nothing more than signed multiplication of their lengths. He then generalized the parallel case to arbitrary line segments using the Pythagorean Theorem. That gave him a definition of the inner product as the signed length-projection of one line onto the other times the other's length. Using his new definition, he then derived the inner product of two free bivectors by treating each as the orthogonal complement of a free vector. He showed that this inner product of bivectors commutes like its corresponding inner product of vectors, which prompted a startling generalization:

"Since the product $a{\times}B$ [the inner product of vector a with bivector B] is not yet defined, we can by analogy [to the aforementioned products] set it equal to $B{\times}a$, whence we have this theorem:

The two factors of any given inner product can be interchanged without changing the value of the product."$_{\text{p344}}$ (his italic emphasis)

This is startling because it's wrong. Grassmann is of course free to establish a "theorem" by analogy (an *axiom*, actually) unless it contradicts his other axioms. This one contradicts his outer-product axioms. (They are entangled with his inner-product axioms via his use of complements.) A different analogy would have been more fruitful. Grassmann might have written…

'Since inner product $a{\times}a$ commutes oppositely to outer product $a.a$, we can by analogy set $a{\times}B$ equal to $-B{\times}a$ (because $a.B$ equals $B.a$), whence we have this theorem:

An outer product that commutes positively with a vector induces an inner product that commutes negatively with it, and conversely.'

This would not have been any more rigorous, but it would have been accidentally right rather than accidentally wrong. (This *very* useful complementary commuting property was later exploited by Clifford to combine the outer and inner products into the *geometric product*.) Grassmann finally got the commuting relations between his outer and inner products right in his 1862 book by using generalized orthogonal complements.

This demonstrates that the inner product was under preliminary development in 1846; and not just for Grassmann, but also for his reviewers and even Mobius, who all passed over this '"theorem" in silence. The final part of his essay, introduced as a *Transition from Displacements to Points*, demonstrates that development was actually *very premature*, what a programmer would call *alpha*. Even so, it was extraordinarily clever, well worth a quick look to observe a creator at work. Be aware that the details may elude you on first reading—they cost me a week of recurring Mobius-assisted effort before I was finally able to decipher them. They hinge on a trick familiar from high school algebra:

$$A^2 - B^2 = (A-B)(A+B)$$

This equation is true even when A and B become free vectors, provided the juxtaposed product is replaced by ×, Grassmann's precursory inner product. When applied to vectors that product distributes, scales and commutes like the product of numbers does. Grassmann used this trick to establish two *intuitive and transparent* ***vector*** equations, which he promptly transitioned into two *unintuitive and opaque* ***point*** equations, like so:

His first set of equations uses what we would now call *position vectors* relative to a*r*bitrary point r, namely $a–r$ and $b–r$, which he substituted for A and B in the preceding equation. That ploy gave him *points* a, b, r within his *vectors*, which produces…

$$(a-r)^2 - (b-r)^2 = (a-b)\times(a+b-2r)$$

…where the squares here are inner-products: $p^2 = p\times p$ for vector p (in his notation). This transparent *vector* equation is an identity, valid for a*r*bitrary *point* r. Grassmann therefor considered r irrelevant, and just extricated it!:[16]

$$a^2 - b^2 = (a-b)\times(a+b)$$

This is an opaque 'equation' in ***points*** a and b. What was he thinking? He was thinking that the right side is now the "inner product" of *vector* $a–b$, eventually called p, with *point* $a+b$, eventually called $2c$. "Thus, the inner product of a point magnitude with a displacement is reduced to the difference of two point squares [on the left]."$_{p364}$ If that statement doesn't baffle you (to paraphrase Bohr) then you don't understand it. There is another baffling statement coming up, so let's defer umbrage until then.

Grassmann's second set of equations uses one position vector $a–r$, and one ordinary vector p:

$$(a-r)^2 - p^2 = (a-r-p)\times(a-r+p)$$

$$a^2 - p^2 = (a-p)\times(a+p)$$

The opaque point equation on the bottom is the transparent vector equation on top with r extricated. Its right side is an "inner product" of *point* $a–p$ with *point* $a+p$. Hence, "the inner product of two point magnitudes is equal to the difference $a^2 - p^2$ of the squares of a point a and a displacement p".$_{ibid}$ This is the second bafflement. Let's rassle with both:

What exactly is "*the difference of two point squares*", $a^2 - b^2$ and "*the difference* $a^2 - p^2$ *of the squares of a point a and a displacement p*"? By derivation, those values are the left sides of their corresponding position-***vector*** equations, all *relative to arbitrary point r*. A few sketches with r in various places should convince you that, by judiciously positioning r, you can make the first value *any scalar* you please, and the second value *any positive scalar* you please, and usually a few negative ones as well. (Those same values will of course be acquired by the corresponding right sides so as to maintain equality.) So how can inner products that produce arbitrary values be of any use?

Here is where things get clever: Grassmann was a master at extricating an origin—that is how he had arrived at addition of points in the first place. His strategy then had been to condense a cloud of vector headpoints, whose tails were all anchored to an arbitrary origin, to just one thing. That thing turned out to be either zero, or a free vector, or a weighted point.

His strategy here is nearly identical: He wants to condense an arbitrary sum of inner products—whether of displacements, position-vector "points", or both—to just one thing. Owing to the preceding tricks, such a sum "can be brought to the form…

$$\Sigma\alpha a^2 + \Sigma\mathrm{A} = 0\text{"}_{p365}$$

…where the a_i are position-vector “points” and the A_i are inner products of displacements, namely scalars. Grassmann carefully disciplines this equation so that he can ignore r, like so: He temporarily reinstalls that origin in his “points” a_i causing them to revert to position vectors $a_i–r$. Then he explores the constraints needed to ignore r in various places. Finally, he shows that a so-constrained sum of inner products can be condensed to an inner product of either two displacements $p \times q$, or a displacement and a point $p \times a$, or else two points $a \times b$.

The first kind of inner product, the usual one, bypasses his algebraic trick and produces a particular scalar. The last two are articulated by that trick, so they produce indeterminate scalars depending on r. Grassmann isn’t concerned with their indeterminacy; he is concerned with *invariant locus* under arbitrary r. Such invariants, unhelpfully called *mean magnitudes*, constitute the final salvo in his struggle to ignore r. The local invariant of each $\alpha_i a_i^2$ (aka $\alpha_i(a_i–r)^2$) is weighted point $\alpha_i a_i$ since r is arbitrary. Remarkably, the sum $\Sigma\alpha a^2 + \Sigma A$ itself has a global invariant, namely $\Sigma\alpha a$, as Grassmann’s careful analysis had informed him. He knew well that $\Sigma\alpha a$, as a sum of weighted points, is either zero, a free vector, or a weighted point.

The zero case constitutes the invariant of an inner product of displacements, the usual one. The free-vector case constitutes the invariant of $a^2 – b^2$, namely free vector $a – b$; so it is an inner product of a displacement and a point (peek back). The point case constitutes the invariant of $a^2 – p^2$, namely point a; so it is an inner product of points. (p^2 has no invariant because free vector p can reside anywhere.)

Such invariants are not the *results* of their corresponding inner products. For example, the result of an inner product of displacements is not its zero invariant, but rather a particular scalar. The results of the other two inner products are worthless, being indeterminate scalars depending on r-in-exile. When faced with such an impasse, Grassmann was a master of improvisation; and he concocted a way to use invariants to make the worthless products worthwhile, like so:

Rather than focusing on indeterminate *results*, he focused instead on the equivalence classes of *arguments* generated via a particular invariant: The inner product whose invariant is a free vector has an equivalence class of point arguments that all go thru the same plane perpendicular to the invariant vector. “We therefore call the inner product of a point with a displacement a *planar magnitude*.”$_{\text{p369}}$ The inner product whose invariant is a point has an equivalence class of point-argument pairs having equal but opposite separation around that point. Consequently, that invariant point serves as the midpoint of an arbitrarily oriented diameter of a sphere. So…“we call the multiple inner product of two points a *spherical magnitude*”.$_{\text{p372}}$

Grassmann’s focus here on an equivalence class of inner-product factors is reminiscent of his pioneering focus on a different kind of equivalence class—that of point summands composing a roving free vector. However the two cases are entirely different:

The free-vector case is a forced move, mandated by the algebra. A sum of points nearly always produces a *particular* weighted point. In the peculiar case where the weight is zero and the summand points fail to annihilate each other, then the result vanishes at infinity. In that case the algebra has no choice but to articulate the equivalence class of *existent* summands that generate that particular *non-existent* result. As for ignoring an origin in that case, there is no need: an origin automatically ignores itself in any free vector.

(If you haven't seen this before, try introducing an origin *r* into free vector *a–b* by substituting *a–r* and *b–r* for points *a* and *b*. What does that coalesce to? Here you see Grassmann's original motivation for discarding an origin.)

His origin-discarding extrapolation to an inner-product of position-vector "points" is not legitimate—they produce a scalar that does depend on the location of the discarded origin. Such an *existent* result is simply *indeterminate* and would generate contradictions if it were used. To bypass such contradictions, Grassmann is certainly free to ignore *r* and focus instead on an equivalence class of factors. However, such intentional disregard for contradiction is not mandated by the algebra (to say the least); it is a trick to make things come out the way he wants.

Grassmann's derivations are extremely clever and perhaps even useful in computer graphics. His *planar* and *spherical* improvisations are related to the *flat* and *round* improvisations currently used in the conformal model of the free Geometric Algebra.

He had second thoughts about inner products involving points, and subsequently abandoned them. In fact, he abandoned nearly everything in this hastily written essay. The only ideas that survived intact were the enduring one that *Extension Theory* is superior to Leibniz's *Characteristic*, and the incidental one of orthogonal complements. That idea was further generalized and considerably polished in Grassmann's…

1862 Extension Theory

A reader, after having rassled with Grassmann's *more pleasing* book, may approach this *so dissimilar* one with high hopes: He has had a good decade and a half to polish his innovative algebra to make it cohere. The amply demonstrated danger is that he might present it in a pristine abstract form far removed from its concrete geometric origins. The *Foreword* confirms this apprehension: "I have applied the most rigorous mathematical form we know, the Euclidean, to the present work, and have relegated to the Remarks everything that serves to illustrate or motivate…"

So a reader eager to discover whether Grassmann has finally begun at the beginning, with points, has no choice but to turn to the chapter where they are finally de-abstracted, namely *Applications to* [mere] *Geometry* in the middle of the book. It is quite encouraging, and quite discouraging.

The encouraging part is that a point, E (the origin, presumably from German *Ein*, meaning one), appears immediately, and is a *formal* part of the language. So points, (denoted by capitals in this chapter only) are going to discipline his algebra right from the beginning. This will prevent *informal point products* from arising and becoming confused with *formal point sums*. That confusion pervaded his 1844 book, manifested most conspicuously by his central equation, $[\alpha\beta] = \beta - \alpha$. Such *overt* manifestation of that confusion is absent in his 1862 book, so far as I can tell.

The discouraging part is that free vectors e_1, e_2, e_3 (again from *ein* presumably) also appear immediately, as tho they were primitive. For geometric consistency, free vectors (denoted by small letters here) really should have been *derived* from subtraction of points, which would have made clear that they are ends only, with nothing in between. This is *quite* discouraging because Grassmann retains his original misinterpretation of a free vector as a *filled-in* "straight line of fixed length and direction"$_{p123}$ but no particular location.

This is again going to infect his algebra right from the beginning with colloquial inconsistencies about dimension. It constitutes a lingering *covert* manifestation of Grassmann's initial confusion between an [αβ] product and a β–α sum. Most discouraging, this will again prevent a bound vector from being understood as a filled-in free vector, and on up.

So, Grassmann did not begin at the beginning, as a tidy reader might have hoped. Instead he began everywhere all at once, as a reader eager to start computing might have wanted.

Such a reader immediately encounters something remarkable: Unlike in his first book, Grassmann now uses consistent terminology to make a clear distinction between free and bound. This would be quite encouraging if he had actually used *free* and *bound*, terms that focus on the contrasting geometric properties of *existent* things. Instead he focused on things that *do not exist*, the free sums that have vanished at infinity:

> By an *infinitely distant point* let the direction of a straight line be meant, by an *infinitely distant straight line* all the directions of a plane, and by an *infinitely distant plane* all the directions of space…$_{p131}$

Grassmann's unexpected proliferation of infinities in this book[17] allows him to "…call both simple [unit-weight] and multiple [non-unit-weight] points, as well as displacements [0-weight free vectors], "points" for short, and in particular "the latter I will call 'infinitely distant points'".$_{p132}$ Thereafter he uses *infinite*-versus-*finite* terminology to make the *free*-versus-*bound* distinction. For example: "from a finitely distant point and three displacements…"$_{p134}$ "If *b* and *c* are *infinitely distant*, that is displacements…"$_{p135}$ (his italics). And so on.

So here is his *free*-versus-*bound* distinction, version two: *free things are infinite, bound things are finite*. This is once again inconsistent with his algebra.

To repeat for the last time, the algebra does not actually articulate the free sums that have vanished to infinity—they simply have no valid limit. If you insist they do, then you immediately generate the contradiction that each one is infinitely distant from itself—see my *Synopsis*. The algebra articulates the only things available to it, the bound summand bundles that produce such a vanished sum. Those bundles are *very **finite***.

Nevertheless, for the purpose of clear communication, it is relief that Grassmann's new terminology clearly distinguishes free from bound. Moreover, for clear communication, it is also a relief that Grassmann has formalized his bracket notation for extension, which he now calls the *combinatorial product*, and uses consistently. His previous triplicitous notation for extension—brackets, dot, juxtaposition—has finally settled on brackets alone.

Such fresh clarity and consistency induced him to discard some of his previous algebra and interpretations. Most significantly, it induced him to discard inner products of points, and indeed, inner products of bound things in general. This is seen in the section entitled *Inner multiplication in geometry*:

> DEFINITION. For inner multiplication I always take as the original units in space three mutually orthogonal and equally long displacements (e_1, e_2, e_3), and in the plane two of them (e_1 and e_2), and in particular I take the length of those displacements as the unit of length, and [e_1 e_2 e_3] , and in the plane [e_1 e_2] as units of volume and surface area.

> REMARK. Hereby all definitions and theorems depending on the concept of inner multiplication are applicable to geometry.$_{p176}$

In other words, when Grassmann actually *interprets* his inner product, he always applies it to *free extensions* derived from an *orthonormal free basis*. The essence of his quoted "*concept of inner multiplication*" is that an inner product of free *magnitudes* A and B is expressed as…

$$[A|B]$$

…where |B denotes the *supplement* of B, a *generalized orthogonal complement*. The brackets denote *generalized extension*, which is either *progressive*, *regressive*, or for the case at hand, *relative*. Don't let those terms confound you—they all spring from one transparent idea, which I shall explain using Grassmann's terminology:

A person typically works within some *principal domain*, such as physical space articulated by basis vectors e_1, e_2, e_3. The free ceiling for that domain is trivector [e_1 e_2 e_3], which Grassmann had just called "the unit of volume". Such a ceiling gives any other free element a *supplement* whose multiplication with it arrives at the ceiling. For example, the supplement of e_2 is [e_3 e_1], the supplement of [e_2 e_1] is $-e_3$, and the supplement of the *absolute unit*, 1, is [e_1 e_2 e_3]. A few sketches will show that if the basis vectors here are orthonormal, then an ***algebraic*** *supplement* would actually be a ***geometric*** *orthogonal complement*, even for the *absolute unit* 1.[18]

Such a precise connection between algebra and geometry was a primary goal of Grassmann, so he imposed a peculiar nonce ceiling axiom on the algebra:

$$[e_1\ e_2\ e_3] \ = \ 1$$

Yes, he set the ceiling unit equal to the floor unit! That startled me (for the nth time) when I first encountered it because it improperly conflates dimensions. Initially I dismissed it as just another improvisation to get things to come out the way he wants. It *is* such an improvisation, and it *does* improperly conflate dimensions, but it gave him a truly elegant way to articulate his inner product—it implicitly introduces the required metric. That metric begins with the following obvious consequence of his ceiling axiom: "*The outer product of a unit and its supplement is 1, that is…*

$$[E|E] \ = \ 1.\text{''}_{p51} \text{ (his italics)}$$

Of particular relevance is Grassmann's use of ***unit*** here. He had written on page 3 that…

> I define as a *unit* any magnitude that can serve for the numerical derivation of a series of magnitudes, and in particular I call such a unit an *original unit* if it is not derivable from another unit. The unit of numbers, that is *one*, I call the *absolute unit*, all others *relative*.

In other words, an *original unit* is a scalable and summable primitive. That excludes zero but little else. In particular, Grassmann's *preliminary units need not have unit "content"* (other than the number unit). He dwelled on the arbitrary magnitude of his units when he showed how to change basis. So why did he use the misleading term *unit* for things with *non-unit content*?

Likely because he had no need to interrelate their *content* until he had introduced his implicit metric. After doing so, however, his *original units*, such as e_1, e_2, e_3, and his *unit* E just displayed, *now did need to have unit content* in order to arrive at the *absolute unit*, 1, as he just asserted they do. Not only that, his *original units now needed to be mutually perpendicular* for the same reason.

Grassmann almost left that unsaid owing to his enthusiasm for abstraction. Fortunately, towards the end of his foundational abstractions he let it slip out in what one reviewer considered an "astonishing" REMARK about *normals*:

> REMARK. The basis of this [normal] nomenclature lies in geometry. There, if one takes the original units as equally long displacements ["as the unit of length"—see the preceding DEFINITION] orthogonal to one another, *as must always be the case*[54]…$_{p98}$ (my italics)

The [54]superscript (present in the original text) leads you to this editorial comment: "*These words are somewhat astonishing, for the "must" makes no sense; one could equally well choose three arbitrary noncoplanar displacements as units*."$_{p353}$ (my italics)

One could, but then Grassmann's inner product would not have the metric he wanted. His REMARK reveals that after he had equated his free ceiling with the *absolute unit* 1, he was tacitly working in an orthonormal free basis, for which his *units* were not only *unitary*, but also mutually perpendicular. In particular, his just-mentioned "units of volume and surface area" do indeed have *unit content*.

By setting his free ceiling unit to his free floor unit 1, Grassmann gained the opportunity to make extension more informative by literally *recycling* those extensions that exceed the ceiling, which would otherwise vanish. He does this in the section entitled *Product with respect to a principal domain*:

> DEFINITION. If the sum of the orders of two units is less than or equal to the order *n* of the principal domain, then by their *progressive product* I mean their outer product, with the condition that the progressive product of the *n* original units is 1. On the other hand, if the sum of the orders of two units is greater than the order (*n*) of the principal domain, then by their *regressive product* I mean that magnitude whose supplement is the progressive product of the supplements of those units.
>
> I combine the progressive and regressive products under the name products *relative to* a principal domain. For all these products the symbol is the same, that is brackets enclosing the elements of the product.$_{p52}$

In short, a bracketed extension that does not exceed the ceiling is *progressive*, one that does is *regressive*, and one that may or may not is *relative*. The versatile *relative* case was needed to articulate Grassmann's inner product. The geometric interpretation of these various products is daunting, but their algebra is elegant and precise. That algebra, as you saw, was commented on in a way that clearly precludes inner products for bound things; so Grassmann really did abandon his previous intricate inner product of points.

Another abandoned idea, as mentioned, was his vertex interpretation of bound extension. For example, his 1862 book now defines [ABC], as a "*surface element*" (his definitional emphasis), whose content is "the area of the *parallelogram* ABC", which "*lies in the plane* ABC" (my italics to emphasize that, in modern terms, this constitutes a *bound bivector* generated by points A, B, C). Next, he makes the following…

> REMARK: One could also have defined the content of the surface element [ABC] as the area of the triangle ABC. But it will be shown below that then the content of the inner square of a displacement would only be half of the content of this displacement, whereas the two are in agreement with our nomenclature.$_{p144}$

This is Grassmann's typically roundabout way of making a bound bivector have the same *area-magnitude* as its free part has *area-separation*.[19] Under his mature interpretation, extension of an element with a point now *parallel-sweeps* that element to the point rather than *vertex-converging* it there. This gives a bound *n*-vector the same *content*—with no factorial in sight—as its free part, which he no longer calls a divergence. In fact, he now declines to even name a free part:

> REMARK: It is these magnitudes [aU], [ABU], [ABCU], that in the first edition of the Ausdehnungslehre of 1844 I called the *divergences* of the magnitudes [a], [AB], [ABC], and therefore a separate name for them is henceforth made superfluous by the use of the *infinitely distant unit* (U).$_{p161}$

An *infinitely distant unit* U is nothing more than a normalized *free ceiling*. Grassmann engages that ceiling with relative bracket multiplication, which collapses to 1 when it hits the ceiling. Such dimensional collapse then *rebuilds* a *bound* argument as a *free* result. His fresh concern with the free-versus-bound distinction in this *Geometry* chapter is a stark contrast to his preceding abstract development, which neglects it.

That distinction has algebraic consequences, so it should not have been neglected. For example, bound elements cannot participate in many of his abstract operations, his inner product in particular. Grassmann was well aware of that, as you have seen; but he evidently did not want to drag mere *geometric* distinction into his abstractions.

But geometric distinction has always been the only way a reader can understand his abstractions; and it has always been his wellspring for them. So a peek into that wellspring is well worthwhile. He left little doubt from comments in both his books (1844.p185, 1862.p98, p123) that he typically conceived his *n*-geometry in a *free-as-possible* ***bound*** *orthonormal basis*—a basis containing just one point, the origin, and *n*–1 orthonormal free vectors.

Elements derived solely from his free vectors are valid in all of his abstractions. Many of them are also valid when they become bound by his origin; but he declined to indicate which ones.[20] In short, Grassmann's "*infinitely distant*" free elements always obey his abstractions, but they may not when they become "*finitely distant*", meaning bound thru a point by extension with it.

His new "*infinitely distant*" terminology may distinguish free from bound better than "*divergence*" did, but it still needlessly obscures the elegant relation between free and bound. That relation, as *the fundamental theorem of the full algebra*, really should be clarified, not obscured. Grassmann was probably doomed to obscure it so long as he retained his interpretation of free things as being filled in. He did retain that interpretation until he died, as you can see from the three-starred footnote to his original "*tentative*" bracket notation, [αβ], that I promised you, in its entirety:

> ***This notation for the displacement is only tentative, since the appropriate notation in terms of its boundary elements can only be understood once we have learned how to conjoin the elements themselves. (1877) In the *Ausdehnungslehre* of 1862 the notation [αβ] is chosen for the product of the two elements α and β, which, if α and β are points, represents the line segment between α and β and is distinguished from the corresponding displacement in that the latter retains only the length and direction, the former also the position, of the infinite straight line to which the line segment belongs. It is therefore all the more important to keep in mind that the notation [αβ] for the displacement is only a temporary makeshift; the appropriate notation β – α can, according to the principle of presentation, only be given in §99.[21]

A reader gets the feeling, from the long (and rare) 1877 appendage to his 1844 footnote, written months before he died, that Grassmann would have much preferred to revise his "*principle of presentation*" so as to avoid the "*temporary makeshift*" bracket notation for displacements, but there was no time for that. In fact, I got that feeling when I first encountered Grassmann's central equation, [αβ] = β – α, written some 33 years prior. You have already seen the partial text of what he wrote after it; here is the full text:

> These two expressions therefore only represent different notations, and since the former is arbitrary, the latter necessary, we prefer to drop the former, regarded from the beginning as only tentative, in favor of the latter, and henceforth denote by β – α a displacement having β as its final element and α as its initial element.$_{p161}$

A reader, knowing about the *extremely paltry time* available to write this book, and reading between his lines (the only way to comprehend Grassmann), may understand this to mean:

> *I have just discovered the proper way to symbolize displacements, so my old way was wrong. I know it is wrong, but to fix it would require completely rewriting my entire book, which I don't have time for, sorry. But I won't keep using a misleading notation henceforth.*

A reader could sympathize with that. A reader understands that a fertile mind sometimes wanders into underbrush, especially one owned by an overburdened teacher. A reader aspiring to a fertile mind may, in fact, actually look forward to seeing how things would look in corrected notation with corrected concepts, as a possible expository exemplar.

Such a reader is startled to discover, two pages later, the old "*tentative*", "*arbitrary*", "*temporary makeshift*" bracketed extension notation for displacements. For the rest of the book! As mentioned, it is bracket product notation in *The Extent to Which the Displacement can be interpreted as a Product*. Could Grassmann possibly have viewed free vectors as *products* if he had kept seeing *sums* in his notation for them? I think not. Could he have retained his filled-in interpretation of free things if he had time to properly revise his first book? I wonder.

After David Hestenes, Hermann Grassmann taught me more than any author I have ever studied. His curious creativity introduced me to an extraordinarily expressive mathematical language, perhaps deserving to be called "the keystone of the entire structure of mathematics" as he did call it in the *Foreword* to his 1862 book. History, I am guessing, may eventually validate that bold assertion. (History, I am also guessing given our current computational hodgepodge, may dawdle about that for several more centuries.)

Grassmann's exceptionally fertile mind, hampered by the *extremely paltry time* available to deploy it, naturally generated blunders. They were magnificent blunders that force a serious reader to re-create his language, and to generate novel personal blunders. Would that we could all let bygones be bygones and march onward in the creative way that gentle, confident Grassmann did.

His relentless onward march, left mostly unrevised and partially abandoned, has taught an unintended lesson: It has demonstrated over and over again that interpretation drives the symbolism, not the other way around. Misinterpretation can easily drive the symbolism to sterility, and even contradiction. Consequently, we should be as careful about mathematical meaning as we have become about mathematical grammar. More careful.[22]

References

Grass.18??

Hermann Gunther Grassmann, *A New Branch of Mathematics, The Ausdehnungslehre of 1844, and Other works*, Open Court. Translated by Lloyd Kannenberg, 1995. Effectively three books in one: the 1878 edition of the Ausdehnungslehre of 1844, the 1847 Geometrisch Analyse, and a rich sampler of Grassmann's articles on mathematics and physics.

Grass.1862

Hermann Grassmann, *Extension Theory*, American Mathematical Society. Translated by Lloyd Kannenberg, 2000. This was Grassmann's attempt to appeal to mathematicians after his 1844 book had failed to do so.

Endnotes

[1] **Grass.1844** Often translated as *Linear Extension Theory*. However, Grassmann's 1800's German *Lineale* actually referred to *lines*, not 1900's English *Linear* operations. His intent was that *Lineale* would distinguish this book from a planned *Angular Extension Theory* that he never wrote.

[2] **Grass.1862** pxiv.

[3] **Grass.1844** p11. "I soon realized that I had come upon the domain of a new science, of which geometry itself is only a special application. … there must be a branch of mathematics that yields in a purely abstract way laws similar to those that in geometry seem bound to space. … The essential advantages of this [purely abstract] approach are, in regard to form, that all principles expressing perceptions of space are now entirely omitted… and in regard to content, that the limitation to three dimensions is absent."

[4] Grassmann later discovered such algebraic bondage as you shall see in detail. It gives line segments a moorage from which to be extended sideways. If line segments really were free to move sideways, then they would have no well-defined moorage from which to be extended to a point. To appreciate the force of this argument, descend one dimension to points that are free to roam around. Would their extension be well-defined?

[5] Ibib.p154. "I associate the concept of the elementary magnitude with the solution of a simple problem by which I first arrived at this concept…" and you are off toward points.

[6] Ibid.p156. *Apologetically*, because *weight* is a concrete term! He justified it in a footnote on this page as being already "used in an abstract sense" in probability theory.

[7] Ibid.p157. He expressed this with typical mind-numbing abstraction: "*An elementary assembly with zero weight deviates from any two elements equally, and an elementary assembly deviating from two elements equally has zero weight and deviates from all elements equally.*" (His italics) His "*elementary assembly with zero weight*" is a cloud of headpoints that condenses to a roving subtraction of two unit points.

[8] gary-harper.com/ Go to the *Geometric Algebra* page.

[9] Be forewarned that Grassmann's derivation implacably confuses dimension because it *presupposes* that free vectors are directed *line segments*. If you begin with that misconception (as you may already have) you will find it very difficult to correct later. That was Grassmann's lifelong affliction, a disorder we all inherited, and my own bane for many years.

[10] I am convinced that Grassmann inserted this footnote after he had discovered the roving point subtraction that behaves like he thought his point extensions should. I further believe that the "*tentatively*" in "We *tentatively* represent the displacement … as [αβ]" was also inserted at that time. Here is why that seems likely: Bracketed [αβ] was used in Grassmann's second book for *extended* (bound) vectors exclusively. Displacements, meaning *free* vectors, were expressed there with point subtraction whenever their genesis from points was needed. Clinching evidence is provided by the long 1877 appendage to this footnote, to be quoted at the end of this paper.

[11] Because point subtraction is nowhere mentioned in this section, Grassmann may have written it before he had decided to drop "*arbitrary*" [αβ] in favor of "*necessary*" β–α. After he did so decide, he perhaps did not have time to revise the many sections his decision modified, such as this one. Or perhaps he just overlooked this section after his epiphany. In any case, he let bygones be bygones and marched onward.

[12] It does so atypically, not by ascent to a higher space as with bound elements, but by *filling in* the free part within the space it already inhabits.

[13] Peano acquired the vertex interpretation from Grassmann's first book, but he understood the fundamental theorem quite well: he called free parts *vectors*, *bivectors* and *trivectors* rather than *divergences*. We would now refine Peano's terms with the prefix *free*, but that would likely have seemed redundant to him because Latin *vector* meant *carrier* in his time, which already connotes *free*. See Giuseppe Peano, *Geometric Calculus*, 1888, translated and annotated by Lloyd Kannenberg, republished by Birkhauser in 1997.

[14] …when it should have been pedaling back.

[15] Clifford seems to have been the only one able to do that effortlessly, likely because he had anticipated many of Grassmann's ideas. Others were also able to decipher Grassmann, most notably Whitehead and Peano, but their comments on the endeavor left little doubt that it was far from easy. It was not easy for me either, and I only partially succeeded; and that only because I had time and enthusiasm to mine Grassmann's ideas for some twenty years. Many others did not have those luxuries, and failed, Hamilton and Gibbs most conspicuously.

[16] Yes, he just extricated it!

[17] His 1844 book has just one mention of infinity, a dismissive one in a footnote to his new β–α notation. He comments that this subtraction may be interpreted as "an infinitely distant element with weight zero, *provided one admits as valid division by zero*."$_{\text{p161}}$ (my italic emphasis) He was evidently unwilling to admit such division in this book—it was never mentioned again.

[18] In this book Grassmann recognizes scalars, like the *absolute unit* 1, to be full-fledged geometric elements having *order zero*, something he did not do in his first book.

[19] *Separation* for free elements versus *magnitude* for bound elements are my terminology, needed for a formal distinction between free and bound. Grassmann's *deviation* in his 1844 book made a distinction somewhat like my *length-separation*; but he abandoned that term in his 1862 book.

[20] He couldn't have—his foundational "*magnitudes of first order*" are either free vectors or points, indiscriminately. They should not have been so-conflated, even abstractly: *non-composite* points and *intrinsically composite* free vectors are *dimensionally distinguished* by weight, or its absence, respectively. They must be so distinguished: they have *dimensionally distinct* habitats—see my *Synopsis*, or *Playing with Geometric Algebra*.

[21] **Gras.1844** p48. His "(1877)" introduces the long appendage of that year.

[22] The rising popularity of so-called "Grassmann Numbers" is a dismaying trend in the opposite direction: They are defined solely by their symbol distinctions and neg-commuting properties, with utter disregard for their geometric meaning; with astounding indifference to Grassmann's actual algebra. They constitute Geometric Algebra stripped of its geometry and eviscerated of its most interconnected algebra. Their use displays an ignorance of the geometry of *n*-vectors in the same way that use of complex numbers displays an ignorance of the geometry of bivectors. Do you see what I mean about the *current computational hodgepodge*?